On Nullsystems in Space of Five Dimensions and their Relation to Ordinary Space.

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The investigations contained in the following pages are chiefly concerned with certain nullsystems in hyperspace. The reason for the special interest in the case $n = 5$ is obvious from the fact that the geometry of point-manifoldnesses in such a space becomes by means of Lie's transformation

$$x_1 = \frac{P_1}{2}, \quad x_2 = X_1, \quad x_3 = \frac{P_2}{2}, \quad x_4 = X_2, \quad x_5 + x_1x_2 + x_3x_4 = X_3,$$

a geometry of surface-elements in ordinary space just as the geometry of ordinary space by the analogous transformation for $n = 3$ employed by Lie in his "Geometrie der Berührungstransformationen" becomes the geometry of line-elements in the plane.

Closely associated with a reduced nullsystem

$$x_2dx_1 - x_1dx_2 + x_4dx_3 - x_3dx_4 + dx_5 = 0,$$

is a complex whose lines are lines of this system and at the same time satisfy the Monge equation

$$dx_1dx_2 + dx_3dx_4 = 0.$$

I have called this an *asymptotic complex* owing to the close relation which exists between it and the linear tangents along asymptotic lines on a surface in ordinary space.

The paper has been divided into three parts. In the first, the general nullsystem in n-dimensional space (n odd) has been derived by means of a Euclidian motion and reduced to its simplest form. Part II discusses the transformation of lines of the nullsystem in five-dimensional space into certain configurations of surface-elements in ordinary space. The problem to find all the two-dimensional surfaces in M_5 whose coordinate lines (u) and (v) belong to an asymptotic

complex, is next taken up, and it is shown that its solution depends on the integration of a differential equation of the second order with equal invariants. Certain applications of asymptotic complexes to surface theory are also given. Finally, in the third part, the question of invariance of the nullsystem and the asymptotic complex, when subjected to projective, and in particular to Euclidian transformations, has been treated from the standpoint of Lie's group-theory. Certain theorems concerning contact-transformations in ordinary space and theorems concerning the mobility of a nullsystem have been obtained.

I. We shall define an infinitesimal motion in the space M_n as an infinitesimal point-transformation that does not alter the distance between two consecutive points. Let there be given the following system of equations:

$$\delta x_i = \xi_i(x_1, x_2, \ldots, x_n)\,\delta t, \qquad (i = 1, 2, \ldots, n), \tag{1}$$

defining an infinitesimal transformation in time δt. In order that the distance shall remain invariant, these equations must satisfy the following relations:

$$\delta(dx_1^2 + dx_2^2 + \ldots\ldots dx_n^2) \equiv 0,$$

or, what is the same thing,

$$dx_1\,\delta dx_1 + dx_2\,\delta dx_2 - \ldots\ldots dx_n\,\delta dx_n = 0.$$

Now since
$$dx_i\,\delta dx_i = dx_i\,d\delta x_i,$$

we get the following conditions:

$$dx_1\,\delta\xi_1 + dx_2\,\delta\xi_2 + \ldots\ldots dx_n\,\delta x_n = 0, \tag{2}$$

but we have also

$$\delta\xi_i = \frac{\partial\xi_i}{\partial x_1}dx_1 + \frac{\partial\xi_i}{\partial x_2}dx_2 + \ldots\ldots + \frac{\partial\xi_i}{\partial x_n}dx_n \; (i = 1, 2, \ldots\ldots, n)$$

and on substituting these values of $\delta\xi_i$ in (2) and equating to zero the coefficients of dx_i^2 and $dx_i\,dx_k$, we obtain the following relations:

$$\frac{\partial\xi_1}{\partial x_1} + \frac{\partial\xi_2}{\partial x_2} = \ldots\ldots = \frac{\partial\xi_n}{\partial x_n} = 0, \tag{3}$$

$$\frac{\partial\xi_i}{\partial x_k} + \frac{\partial\xi_k}{\partial x_i} = 0, \quad i \nless k; \tag{4}$$

from (3) it follows that the functions $\xi_1, \xi_2 \ldots \xi_n$ do not contain the variables $x_1, x_2 \ldots x_n$ respectively. Differentiating the equations (4) partially with respect to x_k we obtain

$$\frac{\partial}{\partial x_k}\frac{\partial \xi_i}{\partial x_k} + \frac{\partial}{\partial x_k}\frac{\partial \xi_k}{\partial x_i} = 0 .$$

Now since the second term on the left hand side is equal to zero, we have

$$\frac{\partial^2 \xi_i}{\partial x_k^2} = 0$$

which shows that the ξ_i's are linear in the x's. The equations (1) therefore reduce to the following

$$\left.\begin{array}{l} \dfrac{\delta x_1}{\delta t} = \quad p_{12}x_2 + p_{13}x_3 + \ldots + p_{1n}x_n \qquad + c_1, \\ \dfrac{\delta x_2}{\delta t} = -p_{12}x_1 + p_{23}x_3 + \ldots + p_{2n}x_n \qquad + c_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dfrac{\delta x_n}{\delta t} = -p_{1n}x_1 - p_{2n}x_2 - \ldots - p_{n-1\,n}x_{n-1} + c_n, \end{array}\right\} \qquad (5)$$

in which p_{ik}, are arbitrary constants.

From the form of the above system we observe that *the most general infinitesimal motion in space of n dimensions having a distance-invariant*

$$= \sqrt{dx_1^2 + dx^2 + \ldots + dx_n^2} = const.$$

is made up of a rotation about the origin transforming the hypersphere

$$x_1^2 + x_2^2 + \ldots + x_n^2 = const.$$

into itself, and n translations along the n axes respectively. In fact, if we let

$$c_1 = c_2 = \ldots = c_n = 0$$

and then multiply the first equation by x_1, the second by x_2 and so on, we obtain

$$x_1 \frac{\delta x_1}{\delta t} + x_2 \frac{\delta x_2}{\delta t} + \ldots + x_n \frac{\delta x_n}{\delta t} = 0$$

or,

$$x_1^2 + x_2^2 + \ldots + x_n^2 = \text{const.}$$

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We shall now consider the ∞^n linear spaces

$$D_1x_1 + D_2x_2 + \ldots + D_nx_n = C, \qquad (6)$$

whose n direction-cosines are proportional to D_1, $D_2 \ldots . D_n$ respectively. Among all these spaces does there exist one which is translated parallel to itself when subjected to the infinitesimal motion (5)? If so it is clear that the following condition must be satisfied

$$D_1 \frac{\delta x_1}{\delta t} + D_2 \frac{\delta x_2}{\delta t} + \ldots . + D_n \frac{\delta x_n}{\delta t} = 0,$$

which, by the use of the equations (5), give rise to the following set of equations:

$$\left.\begin{array}{ll} D_2 p_{12} + D_3 p_{13} + \ldots . + D_n p_{1n} & = 0, \\ -D_1 p_{12} + D_3 p_{25} + \ldots . + D_n p_{2n} & = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ -D_1 p_{1n} - D_2 p_{2n} - \ldots . - D_{n-1} p_{n-1n} & = 0. \end{array}\right\} \tag{7}$$

We shall now consider two cases: n even and n odd. 1°. n even. In this case the determinant of the above system, being a zero-axial skew-determinant of even order, is equal to the square of a certain integral function of the elements (the so-called Pfaffian function), and since the p's are supposed to be all independent of each other, the determinant does not vanish. Hence, there exist in general no values of the D's which will satisfy the above equations, so that *in space of even dimensions no parallel motion of a linear space is possible in general.*

Remark. This statement ceases to be true if the determinant of the p's vanish; but in this case the motion is no longer general; thus for $n = 2$ we have a translation along the axes.

2°. n odd. In this case the skew-determinant vanishes identically; the ratios $\frac{D_1}{D_n}, \frac{D_2}{D_n}, \ldots, \frac{D_{n-1}}{D_n}$ may therefore be found and the problem is possible, provided, of course, not all the minors of order $n-1$ vanish. Having determined the D's to within a factor of proportionality by solving the system (7), we shall next consider all the points in the space M_n moving perpendicular to the space (6). For all such points $\delta x_1, \delta x_2, \ldots, \delta x_n$ must be proportional to the direction-cosines of the linear space; that is, we must have

$$\left.\begin{array}{ll} p_{12}x_2 + p_{13}x_3 + \dots + p_{1n}x_n + c_1 & = \rho D_1, \\ -p_{12}x_1 + p_{23}x_3 + \dots + p_{2n}x_n + c_2 & = \rho D_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ -p_{1n}x_1 - p_{2n}x_2 - \dots - p_{n-1\,n}x_{n-1} + c_n & = \rho D_n. \end{array}\right\} \tag{8}$$

Multiplying these equations by $D_1, D_2, \dots, D_n$ respectively and adding, we obtain

$$c_1D_1 + c_2D_2 + \dots + c_nD_n = \rho\,[D_1^2 + D_2^2 + \dots + D_n^2],$$

which equality determines the factor of proportionality ρ, provided we have

$$D_1^2 + D_2^2 + \dots + D_n^2 \neq 0.$$

The system (8) is thus seen to contain only $n - 1$ independent equations and n unknowns, and represents, therefore, a one-dimensional manifoldness or a straight line. This line will, by the infinitesimal motion (5), be transformed into itself and is, therefore, *the invariant line.**

Let us now choose this invariant line as our x_n-axis, and let all the other axes be perpendicular to it. Since the x_n-axis is invariant, we must have $\delta x_1 = \delta x_2 = \dots = \delta x_{n-1} = 0$ for $x_1 = x_2 = \dots = x_{n-1} = 0$, that is,

$$p_{1n} = p_{2n} = p_{3n} = \dots = p_{n\,n-1} = c_1 = c_2 = \dots = c_{n-1} = 0$$

and our system takes the form

$$\left.\begin{array}{ll} \dfrac{\delta x_1}{\delta t} = & p_{12}x_2 + p_{13}x_3 + \dots + p_{1\,n-1}x_{n-1}, \\ \dfrac{\delta x_2}{\delta t} = & -p_{12}x_1 + p_{23}x_3 + \dots + p_{2\,n-1}x_{n-1}, \\ \dots\dots\dots & \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots & \dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{\delta x_n}{\delta t} = & c_n. \end{array}\right\} \tag{9}$$

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* If $\sum_1^n D_n^2 = 0$, the space $\sum_1^n {}_n x = \text{const.}$ will be tangent to the element-hypercone $dx_1^2 + dx_2^2 + \dots + dx_n^2 = 0$. In fact, in order that the equations $\sum D_n dx_n = 0$ and $\sum dx_n^2 = 0$ shall have a common solution, we must have $dx_1 : dx_2 : \dots : dx_n = D_1 : D_2 : \dots : D_n$; that is, $D_1^2 + D_2^2 + \dots + D_n^2 = 0$. This is therefore the condition for minimal lines; in particular, the invariant line (8) can be no such line.

Of these equations the first $n-1$ satisfy the relation

$$x_1\delta x_1 + x_2\delta x_2 + \ldots\ldots + x_{n-1}\delta x_{n-1} = 0,$$

or

$$x_1^2 + x_2^2 + \ldots\ldots + x_{n-1}^2 = \text{const};$$

hence we obtain the following

THEOREM.—*The most general infinitesimal motion in M_n (n odd) may be decomposed into two distinct motions: a motion leaving the origin at rest and a translation parallel to the invariant axis.*

The above system (9) can be simplified still further. Starting with the original system we shall apply a purely analytical method using the following orthogonal transformation of coordinates

$$\left.\begin{array}{l} x_1 = a_{11}x_1' + a_{12}x_2' + \ldots\ldots + a_{1n}x_n' + k_1, \\ x_2 = a_{21}x_1' + a_{22}x_2' + \ldots\ldots + a_{2n}x_n' + k_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n = a_{n1}x_1' + a_{n2}x_2' + \ldots\ldots + a_{nn}x_n' + k_n, \end{array}\right\} \tag{10}$$

in which the determinant $|a_{11}, a_{22}, \ldots\ldots, a_{nn}| = 1$ and also

$$\left.\begin{array}{l} \sum_{s=1}^{s=n} a_{1s}^2 = 1, \sum_{s=1}^{s=n} a_{2s}^2 = 1, \ldots\ldots, \sum_{s=1}^{s=n} a_{ns}^2 = 1 \\ \sum_{t=1}^{t=n} a_{kt}a_{st} = 0, \quad \left[\begin{array}{l} k = 1, 2, \ldots\ldots, n \\ s = 1, 2, \ldots\ldots, n \\ k \neq s \end{array}\right]; \end{array}\right\} \tag{11}$$

we shall also put

$$P_{ik}^{st} = \begin{vmatrix} a_{is} & a_{it} \\ a_{ks} & a_{kt} \end{vmatrix}, \quad \left[\begin{array}{l} i, k = 1, 2 \ldots\ldots, n \\ s, t = 1, 2 \ldots\ldots, n \end{array}\right]$$

from which it follows that $P_{ik}^{st} = -P_{ik}^{ts}$, $P_{ii}^{st} = 0$ and $P_{ik}^{ss} = 0$.

The system (10) contains $n^2 + n$ parameters all of which are not independent since there exist $\frac{n(n-1)}{2} + n$ relations between them. There remain then $\frac{n(n+1)}{2}$ independent parameters which are functions of the n^2 direction-cosines and of the k's. Introducing now the new coordinates we have

$$\delta x_1' = \sum_1^n a_{s1}\,\delta x_s\,,$$

$$\delta x_2' = \sum_1^n a_{s2}\,\delta x_s\,,$$

$$\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots$$

$$\delta x_n' = \sum_1^n a_{sn}\,\delta x_s\,.$$

Substituting on the right side the values of δx_s, introducing at the same time the values of $x_1, x_2, \ldots, x_n$ obtained from (10), we obtain, after some easy reductions,

$$\left.\begin{aligned}
\frac{\delta x_1'}{\delta t} &= \sum_{i,\,k}^{1\ldots n} p_{ik} P_{ik}^{12}\, x_2' + \sum p_{ik} P_{ik}^{13}\, x_3' + \cdots + \sum p_{in} P_{ik}^{in}\, x_n' + C_1, \\
\frac{\delta x_2'}{\delta t} &= -\sum p_{ik} P_{ik}^{12} x_1' + \sum p_{ik} P_{ik}^{23}\, x_3' + \cdots + \sum p_{ik} P_{ik}^{2n}\, x_n' + C_2, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{\delta x_n'}{\delta t} &= -\sum p\;\; P_{ik}^{1n}\, x_1' + \sum p_{ik} P_{ik}^{2n} x_2' + \cdots - \sum p_{ik} P_{ik}^{n-1\,n} x_{n-1}' + C_n,
\end{aligned}\right\} \quad (12)$$

in which $i \neq k$. The C's may be expressed by the following system of equations:

$$C_1 = \sum_1^n a_{s1}\,\frac{\delta k_s}{\delta t}\,,$$

$$C_2 = \sum a_{s2}\frac{\delta k}{\delta t}\,,$$

$$\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots$$

$$C_n = \sum a_{sn}\,\frac{\delta k_s}{\delta t}\,,$$

where $\frac{\delta k_s}{\delta t}$ stands for what is obtained by substituting for $x_1, x_2 \ldots\ldots x_n$ the

quantities $k_1, k_2 \dots . k_n$ respectively in the original system for the motion. Solving, we find,

$$\frac{\delta k_1}{\delta t} = C_1 a_{11} + C_2 a_{12} + \dots . + C_{1n} a_{1n},$$
$$\frac{\delta k_2}{\delta t} = C_1 a_{21} + C_2 a_{22} + \dots . + C_{2n} a_{2n},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\frac{\delta k_n}{\delta t} = C_1 a_{n1} + C_2 a_{n2} + \dots . + C_n a_{nn}.$$

The C's may now be disposed of by putting all of them equal to zero except one, say C_n. We have then

$$\left.\begin{array}{ll} k_2 p_{12} + k_3 p_{13} + \dots . + k_n p_{1n} + c_1 & = C_n a_{1n}, \\ -k_1 p_{12} + k_3 p_{23} + \dots . + k_n p_{2n} + c_2 & = C_n a_{2n}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ -k_1 p_{1n} - k_2 p_{2n} - \dots . - k_{n-1} p_{n-1\,n} + c_n & = C_n a_{nn}, \end{array}\right\} \quad (13)$$

In order that this system may be satisfied for finite values of the k's, we must have

$$c_1 D_1 + c_2 D_2 + \dots . c_n D_n = C_n \left[a_{1n} D_1 + a_{2n} D_2 + \dots . + a_{nn} D_n\right],$$

where the D's are the minors of the elements of the first column in the above-mentioned skew-symmetrical determinant of the p's. We may now dispose of the a's by putting

$$a_{1n} = \frac{D_1}{\sqrt{D_1^2 + D_2^2 + \dots . + D_n^2}},$$
$$a_{2n} = \frac{D_2}{\sqrt{D_1^2 + D_2^2 + \dots . + D_n^2}}, \dots a_{nn} = \frac{D_n}{\sqrt{D_1^2 + D_2^2 + \dots . + D_n^2}}, \quad (14)$$

and since by hypothesis $D_1^2 + D_2^2 + \dots . + D_n^2 \neq 0$, we obtain the following value for C_n

$$C_n = \frac{c_1 D_1 + c_2 D_2 + \dots . + c_n D_n}{\sqrt{D_1^2 + D_2^2 + \dots . + D_n^2}}.$$

The system (13) shows that the point $(k_1, k_2, \dots . , k_n)$ must be situated on a

straight line which is in fact nothing but the invariant axis obtained before, as is easily verified by comparing the system (13) with (8); we also observe that $C_n = \rho\sqrt{D_1^2 + D_2^2 + \ldots\ldots + D_n^2}$.

If now we introduce the values of $a_{1n}, a_{2n} \ldots\ldots a_{nn}$ given by equations (14) into the equations (12) and reduce by means of (11), putting also at the same time $\Sigma p_{ik} P_{ik}^{12}\delta t$ equal to a new δt, we obtain the following system

$$\left.\begin{aligned}
\frac{\delta x_1}{\delta t} &= \quad x_2 + \frac{\sum p_{ik} P_{ik}^{13}}{\sum p_{ik} P_{ik}^{12}} x_3 + \ldots\ldots + \frac{\sum p_{ik} P_{ik}^{1\,n-1}}{\sum p_{ik} P_{ik}^{12}} x_{n-1}, \\
\frac{\delta x_2}{\delta t} &= -x_1 + \frac{\sum p_{ik} P_{ik}^{12}}{\sum p_{ik} P_{ik}^{12}} x_3 + \ldots\ldots + \frac{\sum p_{ik} P_{ik}^{2\,n-1}}{\sum p_{ik} P_{k}^{12}} x_{n-1}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\frac{\delta x_n}{\delta t} &= C_n = \rho\sqrt{D_1^2 + D_2^2 + \ldots\ldots + D_n^2},
\end{aligned}\right\} \qquad (12')$$

which is of the same form as the system (9). Since the direction-cosines $a_{1n}, a_{2n}, \ldots\ldots, a_{nn}$ have already been determined, there remain $n^2 - n$ with $2(n-1) + \frac{(n-1)(n-2)}{2}$ relations between them, so that there are in the system $\frac{(n-1)(n-2)}{2}$ essential parameters,* or one more than the number of coefficients in the system (12′), not counting C_n. We are, therefore, able to dispose of these at will, provided we do not make the determinant of the system vanish. We may thus put

$$\frac{\sum p_{ik} P_{ik}^{37}}{\sum p_{ik} P_{ik}^{12}} = 1, \quad \frac{\sum p_{ik} P_{ik}^{36}}{\sum p_{ik} P_{ik}^{12}} = 1, \ldots\ldots \frac{\sum p_{ik} P_{ik}^{n-2,\,n-1}}{\sum p_{ik} P_{ik}^{12}} = 1,$$

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* The problem of expressing the n^2 direction-cosines in terms of $\frac{n \cdot n-1}{2}$ essential parameters has been solved by Cayley, Crelle, 32 (1846). See also Klein, "Nicht Euclidische Geometrie," II, p. 109 (Vorlesungshefte).

and all the other coefficients equal to zero. We then obtain the following simple equations

$$\left.\begin{array}{llll}\dfrac{\delta x_1}{\delta t} = x_2 \;, & \dfrac{\delta x_2}{\delta t} = -x_1 \;, & \dfrac{\delta x_3}{\delta t} = x_4, & \dfrac{\delta x_4}{\delta t} = -x_3, \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ \dfrac{\delta x_{n-2}}{\delta t} = x_{n-1}, & \dfrac{\delta x_{n-1}}{\delta t} = -x_{n-2}, & \dfrac{\delta x_n}{\delta t} = C_n, & \end{array}\right\} \qquad (12'')$$

and we have the

THEOREM.—*The most general infinitesimal motion in space of n dimensions (n odd) consists of a translation along an axis and* $\frac{n-1}{2}$ *rotations.* We shall call such a motion *an n-dimensional screw motion.**

Following S. Lie's method† we shall define a line-element in the space M_n as a given point and a direction through it, so that a line-element is completely determined when the coordinates of the point and the direction-cosines, or, what is the same thing, quantities proportional to these are given, that is to say, when the quantities $x_1, x_2 \dots x_n, dx_1 : dx_2 : \dots : dx_n$ are given.

An infinitesimal screw-motion will of course also transform the line-elements of which through each point pass ∞^{n-1}. We shall now consider all those line-elements that are perpendicular to the direction through the point when it is subjected to a screw-motion. For all such elements the following relation must hold

$$dx_1\delta x_1 + dx_2\delta x_2 + \dots + dx_n\delta x_n = 0,$$

which becomes after substituting the values of $\delta x_1, \delta x_2, \dots, \delta x_n$ obtained from $(12'')$

$$x_2dx_1 - x_1dx_2 + x_4dx_3 - x_3dx_4 + \dots + x_{n-1}dx_{n-2} - x_{n-2}dx_{n-1} + C_ndx_n = 0. \qquad (15)$$

Since there are ∞^{2n-1} line-elements there will be ∞^{2n-2} of these satisfying the above differential equation.

Let us inquire what lines satisfy the equation. We write any given line in the form

$$x_i = \rho_i x_n + \sigma_i, \quad (i = 1, 2, \dots, n-1) \qquad (16)$$

* For a discussion of infinitesimal motion in the space (x, y, z) see S. Lie, Geom. der Berührungstr. Vol. I, pp. 206–212.

† S. Lie, Ibid. p. 11.

Substituting in the equation (15) we obtain the following relation between the parameters

$$\sum_{i=1}^{i=\frac{n-1}{2}} (\sigma_{2i}\rho_{2i-1} - \sigma_{2i-1}\rho_{2i}) + C_n = 0. \tag{17}$$

Now, since there are ∞^{2n-2} lines, ∞^{2n-3} of these will satisfy the differential equation (15), namely, all those lines whose parameters satisfy the relation (17). We have then the following extension of a well-known theorem in kinematics of three dimensions:

By an infinitesimal n-dimensional screw-motion, there exist ∞^{2n-3} lines whose points move in a direction perpendicular to the direction of the motion.

Through each point in M_n pass ∞^{n-3} such lines which aggregate we may call a *pencil;* of such pencils, there are in M_n ∞^n and the aggregate of all ∞^{2n-3} lines we shall call a *Nullsystem.*

II.

1. The case $n = 5$ is of special interest, as the following development will show. Our differential equation reduces to

$$x_2dx_1 - x_1dx_2 + x_4dx_3 - x_3dx_4 + C_5dx_5 = 0.$$

Putting $C_5dx_5 = dx_5'$, $x_1 = x_1'$, $x_2 = x_2'$, $x_3 + x_3'$, $x_4 = x_4'$, this equation becomes

$$x_2'dx_1' - x_3'dx_2' + x_4'dx_3' - x_3'dx_4' + dx_5 = 0, \tag{1}$$

and the relation (14) reduces to

$$\rho_2\sigma_1 - \rho_1\sigma_2 + \rho_4\sigma_3 - \rho_3\sigma_4 - 1 = 0.$$

Before we proceed any further, we shall introduce a few definitions due to Lie.*

A surface-element in ordinary space consists of a point x_1, x_2, x_3 and a plane passing through it. Since the direction-cosines of the plane are determined by the quantities $p_1 = \dfrac{\partial x_3}{\partial x_1}$, $p_2 = \dfrac{\partial x_3}{\partial x_2}$ and -1, we may consider x_1, x_2, x_3, p_1, p_2 as the coordinates of an element. There exist in space ∞^5 such elements, so that ordinary space may be considered as a five-dimensional manifoldness, if we choose for space-elements all the ∞^5 surface-elements. A family of surface-

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* Lie-Scheffer, Berührungstr., p. 523.

elements may be expressed analytically by an equation

$$F(x_1, x_2, x_3, p_1, p_2) = 0.$$

Two surface-elements are infinitely near each other when the coordinates of one element differ by an infinitesimal amount from those of the other; that is to say, they are determined by the coordinates $(x_1, x_2, x_3, p_1, p_2)$ and $(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3, p_1 + dp_1, p_2 + dp_2, p_3 + dp_3)$. Whenever the point of the latter element lies in the plane of the former, the two elements are said to have *united position* (Vereinigte Lage). The analytical condition for this may easily be proved to be

$$dx_3 - p_1 dx_1 - p_2 dx_2 = 0.$$

An aggregate of surface-elements, in which each element has united position with all the elements infinitely near it, is called an element-manifoldness, or, shortly, an element-M. (Element-Verein).

If, now, we employ the transformation

$$x_1 = \frac{P_1}{2}, \quad x_2 = X_1, \quad x_3 = \frac{P_2}{2}, \quad x_4 = X_2, \quad x_5 + x_1 x_2 + x_3 x_4 = X_3,$$

where X_1, X_2, X_3, P_1, P_2 represent the coordinates of a surface-element in M_3 and x_1, x_2, x_3, x_4, x_5 those of a point in the space M_5, a one-to-one correspondence is established between all the ∞^5 surface-elements in M_3 and the ∞^5 points of M_5 This transformation is due to Lie.*

Equation (1) may now be written

$$d(x_5 + x_1 x_2 + x_3 x_4) - 2x_1 dx_2 - 2x_3 dx_2 = 0, \tag{2}$$

or, in terms of the coordinates of M_3,

$$dX_3 - P_1 dX_1 - P_2 dX_2 = 0. \tag{2'}$$

Hence we conclude

To a point of M_5 satisfying the Pfaffian equation (2) *there corresponds in M_3 a surface-element; and to a point-manifoldness in M_5 there will correspond in M_3 an element-M.*

The simplest kind of element-M in M_3 consists of a point $X_1 = a$, $X_2 = b$, $X_3 = c$ and all the ∞^2 planes passing through it. To this there corresponds in

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* Lie, Theorie der Transformationsgruppen, 2te Absch., p. 521.

M_5 the two-dimensional plane

$$x_2 = a, \quad x_4 = b, \quad x_5 + ax_1 + bx_3 = c.$$

If the element-M be defined by the four equations

$$X_1 = a, \quad X_2 = b, \quad X_3 = c, \quad \phi(P_1, P_2) = 0,$$

we have what is called an element-cone, consisting of a point and ∞^1 planes passing through it which are tangent to the cone having the point as vertex. In the space M_5 we obtain a curve given by the equations

$$x_2 = a, \quad x_4 = b, \quad x_5 + ax_1 + bx_3 = c, \quad \phi(2x_1, 2x_3) = 0.$$

Again, suppose the element-M be defined by the equations

$$\omega_1(X_1, X_2, X_3) = 0, \quad \omega_2(X_1, X_2, X_3) = 0;$$

solving for X_1 and X_2 we obtain

$$X_1 = \xi_1(X_3) \quad X_2 = \xi_2(X_3) \tag{3}$$

and since the differential equation (2′) must be satisfied by all the surface-elements, we must also have

$$1 - \frac{d\xi_1}{dX_3} P_1 - \frac{d\xi_2}{dX_3} P_2 = 0. \tag{4}$$

Equations (3) and (4) define ∞^2 surface-elements of a curve to which in M_5 there corresponds the two-dimensional surface

$$x_2 = \xi_1(x_5 + x_1x_2 + x_3x_4), \quad x_4 = \xi_2(x_5 + x_1x_2 + x_3x_4),$$
$$1 - 2x_1\left[\frac{d\xi_1}{dX_3}\right] - 2x_3\left[\frac{d\xi_2}{dX_3}\right] = 0,$$

where the bracketed derivatives stand for what is obtained after substituting for X_3 its value in terms of the coordinates of M_5. If to the equations (3) and (4) we add a fourth containing besides the X's also P_1 and P_2, say

$$\omega_3(P_1, P_2, X_1, X_2, X_3) = 0,$$

we obtain an element-M containing only ∞^1 surface-elements. Let this equation be put in the form

$$\rho(X_3, P_1, P_2) = 0. \tag{5}$$

Solving (4) and (5) for P_1 and P_2 we put

$$P_1 = \xi_3(X_3), \quad P_2 = \xi_4(X_3); \tag{6}$$

these values of P_1 and P_2 must satisfy (4), that is, we must have

$$1-\frac{\partial\xi_1}{\partial X_3}\xi_3-\frac{\partial\xi_2}{\partial X_3}\xi_4=0,$$

which may always be satisfied for arbitrary values of ξ_1 and ξ_2 by choosing suitable values for ξ_3 and ξ_4. The ∞^1 surface-elements defined by these equations have their planes tangents at each point along the curve. Such an element-M we shall call an *element-band* (Element-Streife). In the space M_5 there corresponds to this element-M a curve.

Suppose, finally, the element-M be defined by a single equation

$$\lambda(X_1, X_2, X_3)=0; \tag{7}$$

solving for X_3, we put

$$X_3=\eta(X_1, X_2),$$

and, since the equation (4) must also be satisfied, we must have

$$P_1=\frac{\partial\eta}{\partial X_1}, \quad P_2=\frac{\partial\eta}{\partial X_2}. \tag{8}$$

The equations (7) and (8) define ∞^2 surface-elements of a surface whose transform in M_5 is a two-dimensional surface. We shall now summarize these results in the following table:

Space M_3.	Space M_5.
(1). $dX_3-P_1dX_1-P_2dX_2=0$.	(1). $dx_5+x_2dx_1-x_1dx_2$ $+x_4dx_3-x_3dx_4=0$.
(2). ∞^2 surface-elements of a point.	(2). A two-dimensional plane.
(3). ∞^1 surface-elements of a point or element-cone.	(3). A plane curve.
(4). ∞^2 surface-elements of a curve.	(4). A two-dimensional surface.
(5). ∞^1 surface-elements of a curve or element-band.	(5). A curve.
(6). ∞^2 surface-elements of a surface.	(6). A two-dimensional surface.

2. We shall now resume the study of the nullsystem in M_5, and we propose to find the element-M in M_3 corresponding to an ensemble of points represented

by all the lines of the nullsystem. For this purpose it will be convenient to put the nullsystem in the form

$$x_i = \rho_i t + \sigma_i, \qquad (i = 1, 2, 3, 4, 5) \qquad (1)$$

in which $\rho_5 = \rho_2\sigma_1 - \sigma_2\rho_1 + \sigma_3\rho_4 - \sigma_4\rho_3$. Introducing the new coordinates from the transformation on page 114, we have

$$\left.\begin{aligned} &\frac{P_1}{2} = \rho_1 t + \sigma_1, \quad \frac{P_2}{2} = \rho_3 t + \sigma_3, \\ &X_1 = \rho_2 t + \sigma_2, \quad X_2 = \rho_4 t + \sigma_4, \\ &X_3 = (\rho_1\rho_2 + \rho_3\rho_4)\, t^2 + 2(\rho_2\sigma_1 + \rho_4\sigma_3)\, t + \sigma_1\sigma_2 + \sigma_3\sigma_4 + \sigma_5. \end{aligned}\right\} \qquad (2)$$

Does this system define an element-M in M_3? To answer this question we proceed as follows: eliminating t from the above equations, we obtain

$$\left.\begin{aligned} &(a)\quad \rho_2 \frac{P_1}{2} = \rho_1 X_1 + \sigma_1\rho_2 - \sigma_2\rho_1, \\ &(b)\quad \rho_2 \frac{P_2}{2} = \rho_3 X_1 + \sigma_3\rho_2 - \sigma_2\rho_3, \end{aligned}\right\} \qquad (3)$$

$$\left.\begin{aligned} &(a)\quad \rho_4 X_1 - \rho_2 X_2 = \sigma_2\rho_4 - \sigma_4\rho_2, \\ &(b)\quad X_3 = \frac{(\rho_1\rho_2 + \rho_3\rho_4)}{\rho_2^2}(X_1 - \sigma_2)^2 + 2\frac{(\rho_2\sigma_1 + \rho_4\sigma_3)}{\rho_2}(X_1 - \sigma_2) + \sigma_1\sigma_2 + \sigma_3\sigma_4 + \sigma_5. \end{aligned}\right\} (4)$$

Multiplying the first equation by ρ_3 and the second by ρ_1 and subtracting we obtain

$$(c)\quad \rho_3 \frac{P_1}{2} - \rho_1 \frac{P_2}{2} - \sigma_1\rho_3 + \sigma_3\rho_1 = 0. \qquad (4)$$

If now equations [4] (a) and (b) define an element-M we must have

$$dX_3 - P_1 dX_1 - P_2 dX_2 \equiv 0$$

by virtue of the above system, that is to say, the equation

$$\rho_2 P_1 + \rho_4 P_2 - \frac{2(\rho_1\rho_2 + \rho_3\rho_4)}{\rho_2}(X_1 - \sigma_2) - 2(\rho_2\sigma_1 + \rho_4\sigma_2) = 0$$

must be satisfied for values of P_1 and P_2 derived from equations [3] (a) and (b) which is easily seen to be the case. Q. E. D. We have then the

THEOREM.—*To the* ∞^1 *points of a line belonging to a nullsystem in* M_5 *there*

corresponds in M_3 by virtue of the transformation

$$x_1 = \frac{P_1}{2}, \quad x_2 = X_1, \quad x_3 = \frac{P_2}{2}, \quad x_4 = X_2, \quad x_1x_2 + x_3x_4 + x_5 = X_3$$

∞^1 surface-elements forming an element-M. This element-M is formed by the curve defined by the equations

$$\rho_4 X_1 - \rho_2 X_2 - \sigma_2\rho_4 + \sigma_4\rho_2 = 0,$$
$$\frac{(\rho_1\rho_2 + \rho_3\rho_4)}{\rho_2^2}(X_1 - \sigma_2)^2 + \frac{2(\rho_2\sigma_1 + \rho_4\sigma_3)}{\rho_2}(X_1 - \sigma_2) + \sigma_1\sigma_2 + \sigma_3\sigma_4 + \sigma_5 - X_3 = 0,$$

and all the ∞^1 tangent planes along it. The element-M is therefore an *element-band.* The curve may be considered as the intersection of a plane parallel to the X_3-axis and a parabolic cylinder parallel to the X_2-axis. Hence we may say

To the ∞^1 points of a line belonging to a nullsystem in M_5 there corresponds in M_3 ∞^1 surface-elements forming an element-band whose point-locus is a parabola having its plane parallel to the X_3-axis.

To a fixed line of the nullsystem there corresponds one and only one element-band, but it does not follow that to an element-band corresponds only one line of the nullsystem. In fact we shall prove that the correspondence between these space-elements is not a one-to-one. Let us first study the element-band considered as a point-locus merely. We write the nullsystem in the modified form

$$\left.\begin{aligned} x_1 &= \rho_1' x_5 + \sigma_1, & x_2 &= \rho_2' x_5 + \sigma_2, \\ x_3 &= \rho_3' x_5 + \sigma_3, & x_4 &= \rho_4' x_5 + \sigma_4, \end{aligned}\right\} \tag{5}$$

which is obtained from (1) by putting $\sigma_5 = 0$ and $\rho_i' = \frac{\rho_i}{\rho_5}$, $(i = 1, 2, 3, 4)$ and then eliminating the parameter t. The condition that (5) shall be a nullsystem must now be written

$$1 = \rho_2'\sigma_1 - \rho_1'\sigma_2 + \rho_4'\sigma_3 - \rho_3'\sigma_4, \tag{6}$$

while the equations (4), (a) and (b), take the form

$$\rho_4' X_1 - \rho_2' X_2 - \sigma_2\rho_4' + \sigma_4\rho_2' = 0,$$
$$\frac{(\rho_1'\rho_2' + \rho_3'\rho_4')}{\rho_2'^2}(X_1 - \sigma_2)^2 + \frac{2(\rho_2'\sigma_1 + \rho_4'\sigma_3)}{\rho_2'}(X_1 - \sigma_2) + \sigma_1\sigma_2 + \sigma_3\sigma_4 - X_3 = 0.$$

There exist in M_3, ∞^5 such parabolae whose parameters may evidently be written

$$K = \frac{\rho_3'}{\rho_4'}, \qquad L = \sigma_2 - \sigma_4 K,$$

$$A = \frac{\rho_1'\rho_2' + \rho_3'\rho_4'}{\rho_2'^2}, \quad B = \frac{2(\rho_2'\sigma_1 + \rho_4'\sigma_3)}{\rho_2'} - 2\sigma_2 A,$$

$$C = \sigma_1\sigma_2 + \sigma_3\sigma_4 - 2\sigma_2\frac{(\rho_2'\sigma_1 + \rho_4'\sigma_3)}{\rho_2'} + \sigma_2^2 A.$$

Consider now any fixed curve, that is, let A, B, C, K and L be fixed quantities and solve the above equations for ρ_1', ρ_2', ρ_3', σ_1, σ_2 and σ_3, taking also account of the relation (6). We obtain the following set of equations:

$$\left.\begin{array}{ll}
\rho_2' = K\rho_4', & \sigma_1 = B + AL + \sigma_4\left[AK + \frac{BK}{2L}\right] + \frac{C}{L}, \\
\rho_1' = \frac{K}{L}\left[AL + \frac{B}{2}\right]\rho_4' - \frac{1}{L}, & \sigma_2 = L + \sigma_4 K, \\
\rho_3' = \frac{K}{L}\left[1 - \rho_4'\frac{KB}{2}\right], & \sigma_3 = -\frac{BK}{2}\left[1 + \frac{\sigma_4 K}{L}\right] - \frac{CK}{L}.
\end{array}\right\} \quad (7)$$

It follows from these equations that to a fixed parabola in M_3 there corresponds not only one, but ∞^2 lines of the nullsystem, a set of ∞^1 lines being obtained by letting ρ_4' vary and another set of ∞^1 lines by letting σ_4 vary. *Each parabola must therefore be considered as a bundle of ∞^2 coincident curves.*

Remark. A system of lines may be chosen from among the lines of the nullsystem in such a way as to make the correspondence between the configurations a one-to-one. We may call such a system a *one-to-one* system; it may be determined by two relations, one between the ρ's and one between the σ's, say

$$\omega_1(\rho_1', \rho_2', \rho_3', \rho_4') = 0, \quad \omega_2(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = 0,$$

subject to the restriction that none of the parameters A, B, C, K, L shall vanish by virtue of the given relations, or become expressible in terms of one or more of the others.

It now remains to investigate the curve itself considered as an element-band. The tangent plane at a fixed point (X_1, X_2, X_3) has for direction-cosines

quantities proportional to P_1, P_2, -1, P_1 and P_2 being determined by the relations

$$\left.\begin{aligned} \rho_2' \frac{P_1}{2} &= \rho_1' X_1 + \sigma_1\rho_2' - \sigma_2\rho_1', \\ \rho_4' \frac{P_2}{2} &= \rho_3' X_2 + \sigma_3\rho_4' - \sigma_4\rho_3'. \end{aligned}\right\} \quad (4)\,(c)$$

Now these equations show that there can only be ∞^1 surface-elements associated with each point on the curve. In fact, expressing ρ_1', ρ_2', ρ_3' in terms of ρ_4' by means of equations (7) and then eliminating this least parameter from the two equations(4)(c), we obtain a relation between P_1 and P_2 depending on the parameters σ_1, σ_2, σ_3 and σ_4, of which the three first are functions of the last by virtue of the equations (7). We may, therefore, conceive of the configuration as consisting of ∞^1 element-bands having a single parabola as point-locus, or, in other words, it will consist of all the ∞^2 surface-elements of the parabola. We may state the results obtained thus:

To all the ∞^7 lines of the nullsystem (1) *in the space M_5 there correspond ∞^5 parabolae formed by the intersection of the planes*

$$\rho_4 X_1 - \rho_2 X_2 = \sigma_2\rho_4 - \sigma_4\rho_2 \quad (8)$$

and the parabolic cylinders

$$X_3 = \frac{\rho_1\rho_2 + \rho_3\rho}{\rho_2^2}(X_1 - \sigma_2)^2 + \frac{2(\rho_2\sigma_1 + \rho_4\sigma_3)}{\rho_2}(X_1 - \sigma_2) + \sigma_1\sigma_2 + \sigma_3\sigma_4 + \sigma_5. \quad (9)$$

These parabolae must be considered each as a bundle of ∞^2 coincident ones having their surface-elements united into a set of ∞^1 element-bands, so that the configuration corresponding to the nullsystem consists of ∞^5 parabolae (8) *and* (9) *and the ∞^5 surface-elements of each parabola.*

The question now suggests itself: How are the ∞^2 lines of the nullsystem corresponding to a single parabola distributed in the space M_5? Introducing the new parameters A, B, C, K, L, ρ_4', σ into our nullsystem it takes the form

$$\left.\begin{aligned} x_1 &= \left[\left(KA + \frac{BK}{2L}\right)\rho_4' - \frac{1}{L}\right]x_5 + B + AL + \frac{C}{L} + \sigma_4\left(AK + \frac{BK}{2L}\right), \\ x_2 &= K\rho_4' x_5 + L + \sigma_4 K, \\ x_3 &= \frac{K}{L}\left(1 - \rho_4' \frac{KB}{2}\right)x_5 - \frac{B}{2}\left(K + \sigma_4\frac{K^2}{L}\right) - \frac{CK}{L}. \\ x_4 &= \rho_4' x_5 + \sigma_4. \end{aligned}\right\} \quad (10)$$

If now we consider ρ_4', σ_4 as variable parameters while the others are fixed, we obtain all the ∞^2 lines of the nullsystem corresponding to a given parabola in M_3. These lines lie in the two-dimensional plane defined by the three flat spaces or lineoids

$$\left.\begin{aligned}
&x_2 - Kx_4 - L = 0\,, \\
&x_3 + \frac{BK^2}{2L}x_4 - \frac{K}{L}x_5 + \frac{BK}{2} + \frac{CK}{L} = 0\,, \\
&x_1 - \left(AK + \frac{BK}{2L}\right)x_4 + \frac{x_5}{L} - B - AL - \frac{C}{L} = 0\,,
\end{aligned}\right\} \tag{11}$$

obtained by eliminating ρ_4' and σ_4 from the system (10). There are in M_5 ∞^5 such planes, corresponding to the ∞^5 parabolae in M_3. Through each point in M_5 pass ∞^2 such planes and hence through each point of M_3 considered as a surface-element there must pass ∞^2 parabolae; (this is also evident from the consideration that there are ∞^2 parabolae in space having their planes parallel to the X_3-axis and being also tangent to a given plane at a given point in the plane.) But we know that through a given point in M_5 pass ∞^3 lines of the nullsystem; hence these lines must be distributed in the ∞^2 planes passing through the point, ∞^1 of them being situated in each plane and passing through the point. We have thus arrived at a definite idea of the grouping of the lines of the nullsystem in M_5, the transformation into ordinary space having furnished us a means of geometrical "Anschauung". The two-dimensional manifoldness (11) in which are situated all the ∞^2 lines of the nullsystem corresponding to a given parabola in M_3 we shall call *a point-M_2*, and we may state the result obtained this:

To all the lines of a point-M_2 in M_5 there corresponds in M_3 a parabola, and conversely, to a parabola in M_3 there corresponds all the lines of a point-M_2.

We may now collect these results in the following table:

Space M_5.	Space M_3.
(1). $dx_5 + x_2 dx_1 - \ldots\ldots = 0$.	(1). $dX_3 - P_1 dX_3 - P_2 dX_2 = 0$.
(2). Point.	(2). Surface-element.
(3). A straight line of the nullsystem.	(3). A parabola considered as an element-band.
(4). All the ∞^2 lines of an element-M_2.	(4). A single parabola considered as ∞^1 united element-bands.
(5). All the ∞^3 lines of the nullsystem passing through a fixed point.	(5). ∞^2 parabolae passing through a fixed point.
(6). All the ∞^2 point-M_2's passing through a fixed point.	(6). ∞^2 parabolae passing through a fixed point.
(7). All the ∞^5 point-M_2's.	(7). All the ∞^5 parabolae.

3. If we choose all the lines of the nullsystem for which $\rho_1\rho_2 + \rho_3\rho_4 = 0$ we obtain the ∞^4 straight lines

$$\left.\begin{aligned} X_1 &= \rho_2 t + \sigma_2 , \\ X_2 &= \rho_4 t + \sigma_4 , \\ X_3 &= 2(\rho_2\sigma_1 + \rho_4\sigma_3)\, t + \sigma_1\sigma_2 + \sigma_3\sigma_4 + \sigma_5 , \end{aligned}\right\} \tag{1}$$

the planes of whose surface-elements are determined by the equations

$$\frac{P_1}{2} = \rho_1 t + \sigma_1 ,$$

$$\frac{P_2}{2} = \rho_3 t + \sigma_3 ,$$

from which we obtain by virtue of the above relations between the ρ's

$$\rho_2 \frac{P_1}{2} + \rho_4 \frac{P_2}{2} = \sigma_1\rho_2 + \sigma_3\rho_4 . \tag{2}$$

Now this relation is identical with the relation (3) of page 117; in fact, multiplying it by ρ_3 and (3) by ρ_2, we get by adding

$$(\rho_1\rho_2 + \rho_3\rho_4)\frac{P_2}{2} \equiv \sigma_1(\rho_2\rho_3 - \rho_2\rho_3) + \sigma_3(\rho_1\rho_2 + \rho_3\rho_4) \equiv 0 .$$

Hence, only one relation exists between P_1 and P_2. We may therefore say:

To all the ∞^6 lines of M_5 satisfying the differential equations

$$\left.\begin{aligned} dx_5 + x_2dx_1 - x_1dx_2 + x_4dx_3 - x_3dx_4 &= 0\,, \\ dx_1dx_2 + dx_3dx_4 &= 0 \end{aligned}\right\} \tag{3}$$

there correspond in M_3 all the ∞^4 lines of that space, together with all the surface-elements of these lines.

Eliminating the parameter t from the equation (1), we obtain

$$X_1 = r_1X_3 + s_1, \quad X_2 = r_2X_3 + s_2\,,$$

where

$$r_1 = \frac{\rho_2}{2(\rho_2\sigma_1 + \rho_4\sigma_3)}\,, \quad r_2 = \frac{\rho_4}{2(\rho_2\sigma_1 + \rho_4\sigma_3)}\,,$$

$$s_1 = \frac{2\sigma_2(\rho_2\sigma_1 + \rho_4\sigma_3) - \rho_2(\sigma_1\sigma_2 + \sigma_3\sigma_4 + \sigma_5)}{2(\rho_2\sigma_1 + \rho_4\sigma_3)}\,,$$

$$s_2 = \frac{2\,\sigma_4(\rho_2\sigma_1 + \rho_4\sigma_3) - \rho_4(\sigma_1\sigma_2 + \sigma_3\sigma_4 + \sigma_5)}{2(\rho_2\sigma_1 + \rho_4\sigma_3)}\,.$$

The complex of lines satisfying the differential equations (3) is thus seen to be a remarkable one, inasmuch as a correspondence exists between all the lines belonging to it and all the lines of ordinary space, these lines being considered as an aggregate of surface-elements. The ∞^6 lines of the complex satisfy a certain geometrical condition; in fact, they are lines belonging to the system of ∞^4 4-dimensional cylinders of the second degree

$$(x - \sigma_1)(x_2 - \sigma_2) + (x_3 - \sigma_3)(x_4 - \sigma_4) = 0$$

parallel to the x_5-axis. A complex of this kind we shall call *asymptotic*. The reason for this name will appear later. Its equations are

$$\left.\begin{aligned} x_1 &= \frac{1}{L}\left(\frac{BK}{2}\rho_4' - 1\right)x_5 + B + \frac{C}{L} + \sigma_4\frac{BK}{2L}\,, \\ x_2 &= K\rho_4'x_5 + L + \sigma_4K\,, \\ x_3 &= \frac{K}{L}\left(1 - \rho_4'\frac{KB}{2}\right)x_5 - \frac{B}{2}\left(K + \sigma_4\frac{K^2}{L}\right) - \frac{CK}{L}\,, \\ x_4 &= \rho_4'x_5 + \sigma_4. \end{aligned}\right\} \tag{4}$$

To a line in M_5 corresponds a certain line in M_3 considered as an element-band, which may be looked upon as a line with a plane passing through it; but to a line in M_3 there will correspond ∞^2 lines of the nullsystem which lie in the two-dimensional manifoldness defined by the equations

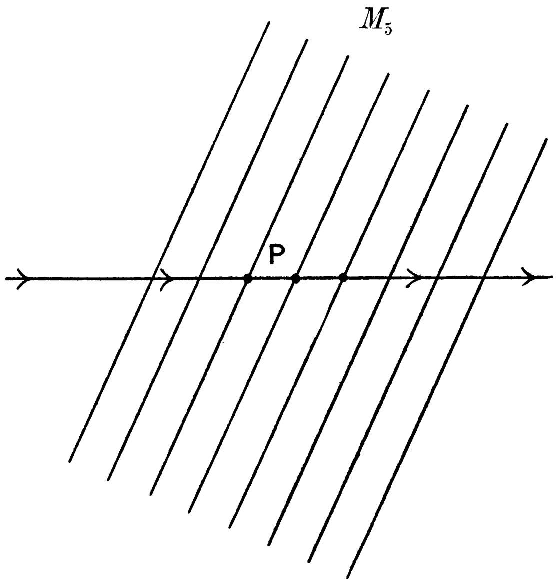
$$\begin{aligned} &x_2 - Kx_4 - L = 0\,, \\ &x_1 - \frac{BK}{2L}x_4 + \frac{x_5}{L} - B - \frac{C}{L} = 0\,, \\ &\frac{x_3}{K} + \frac{BK}{2L}x_4 - \frac{x_5}{L} + \frac{B}{2}\left(1 + \frac{2C}{BL}\right). \end{aligned}$$

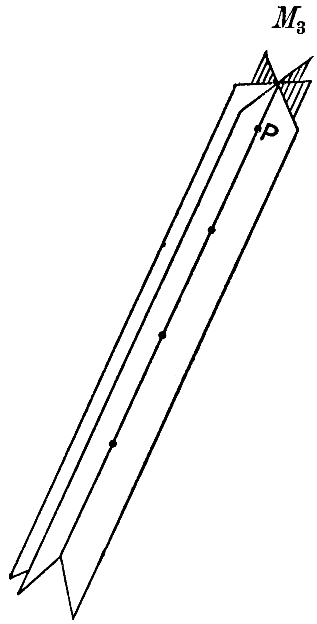
Through a given point in M_5 there pass ∞^1 such point-M_2's, hence, through a fixed point in M_3 pass ∞^1 lines all lying in the plane of the surface-element determined by the coordinates of the fixed point in M_5. To all the ∞^1 lines of the nullsystem passing through a given point and lying in a given point-M_2 corresponds a single line in M_3 passing through a point and lying in a plane whose coordinates P_1 and P_2 are determined by the corresponding values of x_1 and x_3, (the plane of the surface-element). Suppose now that the point moves in any given point-M_2; the corresponding point in M_3 will move along the line, while the plane of the surface-element of the line will turn around the line as an axis. The reason why we only get ∞^1 planes, while there are ∞^2 points in the point-M_2 is explained by the fact that there are ∞^1 points in it that determine the same plane, namely all the points of the point-M_2 for which $x_1 = \frac{P_1}{2} = \text{const.}$ $x_3 = \frac{P_2}{2} = \text{const.}$ These points lie on a certain line whose equations are easily obtained, viz.,

$$\left.\begin{aligned} x_1 &= C_1\,, \\ x_2 &= Kx_4 + L\,, \\ x_3 &= -K\left[C_1 - \frac{B}{2}\right], \\ x_4 &= \frac{2}{BK}x_5 + \left(C_1 - B - \frac{C}{L}\right)\frac{2L}{BK}. \end{aligned}\right\} \tag{5}$$

Now, suppose C_1 a variable parameter; we obtain ∞^1 parallel lines in the point-M_2, to each of which there corresponds in M_3 a different plane. If, then, the point moves along some line cutting this set of parallel lines at some angle, the plane of the surface-element will rotate around the line as an axis and com-

plete one revolution when the point P has returned to its original position. In the same way, if one of the lines (5) move parallel to itself, the plane of the surface-element in M_3 will rotate and resume its original position when the line has returned to its original position. Hence, it follows that there is a one-to-one correspondence between the ∞^1 lines (5) and the ∞^1 planes of the surface-elements of a line in M_3.





The following table will now be convenient as a résumé of the results obtained:

Space M_5.	Space M_3.
(1). $dx_5 + x_2dx_1 - x_1dx_2 + x_4dx^3 - x_3dx_4 = 0$.	(1). $dX_3 - P_1dX_1 - P_2dX_2 = 0$.
(2). $dx_1dx_2 + dx_3dx_4 = 0$.	(2). $dP_1dX_1 + dP_2dX_2 = 0$.
(3). Point.	(3). Surface-element.
(4). All the ∞^6 lines of the complex.	(4). All the ∞^4 lines of M_3 and the ∞^1 surface-elements of each line.
(5). The ∞^2 lines of a point-M_2.	(5). A single line and its ∞^1 surface-elements.
(6). All the ∞^1 point-M_2's passing through a fixed point.	(6). All the ∞^1 lines passing through a fixed point and lying in the plane of the surface-element determined by the point.
(7). All the ∞^1 parallel lines (5) lying in a point-M_2.	(7). A line and its ∞^1 surface-elements.

4. The study of geometrical relations in hyper-space is often very useful for the investigation of geometrical relations in ordinary space. If a correspondence of a higher space with ordinary space can be effected by means of a transformation, we are very often led to interesting properties of space that would otherwise not have been so evident.

In the following we shall show how the study of an asymptotic complex will lead to the study of all surfaces whose asymptotic lines are known.

A curve c_5 in the space M_5 is said to be a curve of the nullsystem whenever the linear tangent at each point of the curve belongs to the nullsystem. The most general curve of the system has the form

$$x_1 = \phi_1(u), \quad x_2 = \phi_2(u), \quad x_3 = \phi_3(u), \quad x_4 = \phi_4(u),$$
$$x_5 = \int (\phi_1\phi_2' - \phi_2\phi_1' + \phi_3\phi_4' - \phi_4\phi_3')\, du.$$

In M_3 we get a curve c_3,

$$X_1 = \phi_2, \qquad X_2 = \phi_4,$$
$$X_3 = \phi_1\phi_2 + \phi_3\phi_4 + \int (\phi_1\phi_2' - \phi_2\phi_1' + \phi_3\phi_4' - \phi_4\phi_3')\, du.$$

At each point of this curve, the plane of the surface-elements is tangent to the curve. To the curve c_5, considered as the envelope of all its tangents, corresponds the curve c_3 considered as the envelope of ∞^1 parabolae. Now, suppose that the nullsystem is asymptotic; we have then in addition the relation

$$\phi_1'\phi_2' + \phi_3'\phi_4' = 0,$$

and our curve must have the form

$$x_1 = \phi_1, \quad x_2 = \phi_2, \quad x_3 = \phi_3, \quad x_4 = -\int \frac{\phi_1'\phi_2'}{\phi_3'}\, du,$$
$$x_5 = \int \left[\phi_1\phi_2' - \phi_2\phi_1' - \phi_3 \cdot \frac{\phi_1'\phi_2'}{\phi_3'} + \phi_3' \cdot \int \frac{\phi_1'\phi_2'}{\phi_3'}\, du \right] du,$$

to which there corresponds the following curve in M_3:

$$X_1 = \phi_2 , \quad X_2 = \phi_4 - \int \frac{\phi_1' \phi_2'}{\phi_3'} du , \quad X_3 = \phi_1 \phi_2 - \phi_3 \int \frac{\phi_1' \phi_2'}{\phi_3'} du + \phi_5 ,$$

where $\phi_5 = x_5$. *The plane of the surface-element at each point of this curve osculates the curve.* In fact, we have

$$\frac{dx_5}{du} - x_1 \frac{dx_2}{du} + x_2 \frac{dx_1}{du} - x_3 \frac{dx_4}{du} - x_4 \frac{dx_3}{du} = \frac{dX_3}{du} - P_1 \frac{dX_1}{du} - P_2 \frac{dX_2}{du} = 0 ,$$

$$\frac{dx_1}{du}\frac{dx_2}{du} + \frac{dx_3}{du}\frac{dx_4}{du} = \frac{dX_1}{du}\frac{dP_1}{du} + \frac{dX_2}{du}\frac{dP_2}{du} = 0 ;$$

of these equations, the first expresses the fact that the plane of the surface-element is tangent to the curve, while the second is nothing but the condition that the tangent plane shall osculate the curve. Q. E. D.

A two-dimensional surface S_2 in M_5 is said to belong to the nullsystem whenever the curvilinear coordinate system $u =$ const., $v =$ const., belongs to the system. Let the system be determined by the equations

$$X_i = \phi_i (u, v). \qquad (i = 1, 2, 3, 4, 5) \tag{1}$$

In order that this system shall belong to the nullsystem, the following conditions must be satisfied:

$$\left.\begin{aligned} \frac{\partial \phi_5}{\partial u} &= \phi_1 \frac{\partial \phi_2}{\partial u} - \phi_2 \frac{\partial \phi_1}{\partial u} + \phi_3 \frac{\partial \phi_4}{\partial u} - \phi_4 \frac{\partial \phi_3}{\partial u} , \\ \frac{\partial \phi_5}{\partial v} &= \phi_1 \frac{\partial \phi_2}{\partial v} - \phi_2 \frac{\partial \phi_1}{\partial v} + \phi_3 \frac{\partial \phi_4}{\partial v} - \phi_4 \frac{\partial \phi_3}{\partial v} . \end{aligned}\right\} \tag{2}$$

Now, since $d\phi_5$ must be a perfect differential, we have

$$\frac{\partial \phi_1}{\partial v}\frac{\partial \phi_2}{\partial u} - \frac{\partial \phi_2}{\partial v}\frac{\partial \phi_1}{\partial u} + \frac{\partial \phi_3}{\partial v}\frac{\partial \phi_4}{\partial u} - \frac{\partial \phi_4}{\partial v}\frac{\partial \phi_3}{\partial u} = 0 . \tag{3}$$

Treating this equation as a linear differential equation in $\frac{\partial \phi_2}{\partial u}$ and $\frac{\partial \phi_2}{\partial v}$, it may be written

$$A \frac{\partial \phi_2}{\partial u} - B \frac{\partial \phi_2}{\partial v} = C ,$$

where
$$A = \frac{\partial \phi_1}{\partial v} , \quad B = \frac{\partial \phi_1}{\partial u} , \quad C = \frac{\partial \phi_4}{\partial v}\frac{\partial \phi_3}{\partial u} - \frac{\partial \phi_3}{\partial v}\frac{\partial \phi_4}{\partial u} .$$

We find, by integrating,

$$\phi_1(u, v) = c_1, \quad \phi_2 = -\int \Big[\frac{C}{B}\Big]_{u=\rho(v,\, c)} dv = \xi(v, c_1) + c_2,$$

where the expression $\Big[\dfrac{C}{B}\Big]$ stands for the value of the function $\dfrac{C}{B}$ after substituting in it for u the value of $\rho(v)$ obtained by solving $\phi_1(u, v) = c_1$ for u. The general integral of (3) is, therefore,

$$\phi_2 = \Phi(\phi_1) + \xi(v, \phi(u, v)),$$

where the function Φ is arbitrary. We have then the following

THEOREM.—*The most general two-dimensional surface in M_5 belonging to a nullsystem is given by the equations*

$$\left.\begin{aligned} x_1 &= \phi_1(u, v), \\ x_2 &= \phi_2(u, v) = \Phi(\phi_1) - \int \Big[\frac{C}{B}\Big]_{u=\rho(v,\, c_1)} dv = \Phi(\phi_1) + \xi(v, \phi_1(u, v)), \\ x_3 &= \phi_3(u, v), \\ x_4 &= \phi_4(u, v), \\ x_5 &= \phi_5(u, v) = \int \frac{\partial \phi_5}{\partial u} du + \frac{\partial \phi_5}{\partial v} dv, \end{aligned}\right\} \tag{4}$$

where

$$\frac{C}{B} = \frac{\dfrac{\partial \phi_4}{\partial v}\dfrac{\partial \phi_3}{\partial u} - \dfrac{\partial \phi_3}{\partial v}\dfrac{\partial \phi_4}{\partial u}}{\dfrac{\partial \phi_1}{\partial u}}$$

and ϕ_5 satisfies the conditions

$$\frac{\partial \phi_5}{\partial u} = \phi_1 \frac{\partial \phi_2}{\partial u} - \phi_2 \frac{\partial \phi_1}{\partial u} + \phi_3 \frac{\partial \phi_4}{\partial u} - \phi_4 \frac{\partial \phi_3}{\partial u},$$

$$\frac{\partial \phi_5}{\partial v} = \phi_1 \frac{\partial \phi_2}{\partial v} - \phi_2 \frac{\partial \phi_1}{\partial v} + \phi_3 \frac{\partial \phi_4}{\partial v} - \phi_4 \frac{\partial \phi_3}{\partial v}.$$

The corresponding surface in M_3 may now be written

$$\left.\begin{aligned} X_1 &= \phi_2(u, v) = \Phi(\phi_1) + \xi(v, \phi_1(u, v)), \\ X_2 &= \phi_4(u, v), \\ X_3 &= \phi_1(u, v)\phi_2(u, v) + \phi_3(u, v)\phi_4(u, v) + \phi_5(u, v). \end{aligned}\right\} \tag{5}$$

To a point $u =$ const., $v =$ const. on the surface (4) there corresponds on (5) a surface-element consisting of a point $u =$ const., $v =$ const. and a plane passing through it. *This plane is tangent to the surface at that point.* In fact, the condi-

tions that the plane shall be tangent to the surface are

$$\left.\begin{aligned}\frac{\partial X_3}{\partial u} - P_1\frac{\partial X_1}{\partial u} - P_2\frac{\partial X_2}{\partial u} = 0,\\ \frac{\partial X_3}{\partial v} - P_1\frac{\partial X_1}{\partial v} - P_2\frac{\partial X_2}{\partial v} = 0,\end{aligned}\right\} \tag{6}$$

and if we calculate the partial derivatives of X_1, X_2 and X_3, as well as also P_1 and P_2 from the equations (5) and substitute in (6), these equations reduce to an identity. The above statement is also true geometrically, since the plane must be tangent to both curves at the point $u =$ const. $v =$ const. We may say then

To all points of a two-dimensional surface in M_5 belonging to a nullsystem there corresponds in M_3 all the surface-elements of a surface.

The converse is also true and may easily be proved.

A two-dimensional surface S_2 in M_5 is said to belong to an asymptotic complex

$$\left.\begin{aligned}dx_5 - x_1dx_2 + x_2dx_1 - x_3dx_4 + x_4dx_3 &= 0,\\ dx_1dx_2 + dx_3dx_4 &= 0,\end{aligned}\right\} \tag{7}$$

whenever the coordinate system (u), (v) belongs to the complex. The functions ϕ_i, $(i = 1, 2, 3, 4, 5)$ must now satisfy the following conditions:

$$\left.\begin{aligned}\frac{\partial\phi_5}{\partial u} = \phi_1\frac{\partial\phi_2}{\partial u} - \phi_2\frac{\partial\phi_1}{\partial u} + \phi_3\frac{\partial\phi_4}{\partial u} - \phi_4\frac{\partial\phi_3}{\partial u},\\ \frac{\partial\phi_5}{\partial v} = \phi_1\frac{\partial\phi_2}{\partial v} - \phi_2\frac{\partial\phi_1}{\partial v} + \phi_3\frac{\partial\phi_4}{\partial v} - \phi_4\frac{\partial\phi_3}{\partial v},\end{aligned}\right\} \tag{8 (a)}$$

$$\frac{\partial\phi_1}{\partial u}\frac{\partial\phi_2}{\partial u} + \frac{\partial\phi_3}{\partial u}\frac{\partial\phi_4}{\partial u} = 0, \quad \frac{\partial\phi_1}{\partial v}\frac{\partial\phi_2}{\partial v} + \frac{\partial\phi_3}{\partial v}\frac{\partial\phi_4}{\partial v} = 0. \tag{8 (b)}$$

Transforming into M_3 we obtain a surface on which (u) and (v) are asymptotic curves. In fact, introducing the coordinates X_1, X_2, X_3, P_1 and P_2 in 8 (a) and 8 (b) we obtain

$$\left.\begin{aligned}\frac{\partial X_3}{\partial u} = P_1\frac{\partial X_1}{\partial u} + P_2\frac{\partial X_2}{\partial u}, \quad \frac{\partial X_3}{\partial v} = P_1\frac{\partial X_1}{\partial v} + P_2\frac{\partial X_2}{\partial v},\\ \frac{\partial X_1}{\partial u}\frac{\partial P_1}{\partial u} + \frac{\partial X_2}{\partial u}\frac{\partial P_2}{\partial u} = 0, \quad \frac{\partial X_1}{\partial v}\frac{\partial P_1}{\partial v} + \frac{\partial X_2}{\partial v}\frac{\partial P_2}{\partial v} = 0,\end{aligned}\right\} \tag{9}$$

of these equations the first two expresses the fact that the plane of the surface elements is tangent to the surface, while the two last are the conditions that this

plane shall osculate the curves (u) and (v), that is (u) and (v) are asymptotic lines.

The problem to find all the surfaces belonging to an asymptotic complex is thus seen to be reduced to the problem of finding all the surfaces in ordinary space on which (u) and (v) are asymptotic lines. For the solution of this problem the reader is referred to Vol. IV of Darboux's "Théorie des Surface," page 20. (Lelieuvre's formulae).

Conversely, the problem to find surfaces on which (u) and (v) are asymptotic lines may from the standpoint of asymptotic complexes be considered equivalent to the problem of finding all the two-dimensional surfaces belonging to such a complex. Treating the problem from this point of view we proceed as follows:*

The function ϕ_i, $(i = 1, 2, 3, 4, 5)$ in addition to satisfying the conditions 8 (a) and 8 (b) must, as we have seen, as a consequence also satisfy the equation

$$\frac{\partial\phi_1}{\partial v}\frac{\partial\phi_2}{\partial u} - \frac{\partial\phi_2}{\partial v}\frac{\partial\phi_1}{\partial u} + \frac{\partial\phi_3}{\partial v}\frac{\partial\phi_4}{\partial u} - \frac{\partial\phi_4}{\partial v}\frac{\partial\phi_3}{\partial u} = 0,$$

which by 8 (b) reduces to the form

$$\left(\frac{\partial\phi_2}{\partial u}\frac{\partial\phi_3}{\partial v} + \frac{\partial\phi_2}{\partial v}\frac{\partial\phi_3}{\partial u}\right)\left(\frac{\partial\phi_1}{\partial v}\frac{\partial\phi_3}{\partial u} - \frac{\partial\phi_1}{\partial u}\frac{\partial\phi_3}{\partial v}\right) = 0.$$

If the second factor vanishes we obtain in M_5 a curve instead of a surface contrary to hypothesis; we may therefore exclude this case and put

$$\frac{\partial\phi_2}{\partial u}\frac{\partial\phi_3}{\partial v} + \frac{\partial\phi_2}{\partial v}\frac{\partial\phi_3}{\partial u} = 0. \tag{10}$$

From 8 (b) and (10) we obtain

$$\left.\begin{aligned}\frac{\partial\phi_1}{\partial u} &= -\frac{\dfrac{\partial\phi_3}{\partial u}}{\dfrac{\partial\phi_2}{\partial u}}\cdot\frac{\partial\phi_4}{\partial u} \equiv -R\frac{\partial\phi_4}{\partial u_4},\\ \frac{\partial\phi_1}{\partial v} &= -\frac{\dfrac{\partial\phi_3}{\partial v}}{\dfrac{\partial\phi_2}{\partial v}}\cdot\frac{\partial\phi_4}{\partial v} = R\frac{\partial\phi_4}{\partial v}.\end{aligned}\right\} \tag{11}$$

* This method was developed by the writer at a time when he had no access to any literature on the subject of asymptotic lines; it is therefore strictly original, and presents the problem of Lelieuvre from an entirely new point of view.

From (10) and (11) we obtain, since $d\phi_1$ and $d\phi_3$ must be exact differentials, the following differential equation which must be satisfied by the functions ϕ_2 and ϕ_4

$$\frac{\partial^2\theta}{\partial u\,\partial v}+\tfrac{1}{2}\frac{\partial}{\partial v}\log R\,.\frac{\partial\theta}{\partial u}+\tfrac{1}{2}\frac{\partial}{\partial u}\log R\,.\frac{\partial\phi_2}{\partial v}=0\,. \tag{12}$$

If two particular solutions ϕ_2 and ϕ_4 can be found the determination of ϕ_1, ϕ_3 and ϕ_5 from (11), (10) and (8) will involve quadratures only. *The problem to find all surfaces on which* (u) *and* (v) *are asymptotic lines is thus reduced to the integration of the differential equation* (18) *which is one of equal invariants and may be reduced to the form*

$$\frac{\partial^2\theta}{\partial u\,\partial v}=h_1\theta\,; \tag{12'}$$

hence the

THEOREM.—*If a surface* $x_i=\phi_i(u,v)$ *in* M_5 *belongs to an asymptotic complex the coordinates* x_2 *and* x_4 *must satisfy a differential equation of the form*

$$\frac{\partial^2\theta}{\partial u\,\partial v}+\tfrac{1}{2}\frac{\partial}{\partial v}\log R\,.\frac{\partial\theta}{\partial u}+\tfrac{1}{2}\frac{\partial}{\partial u}\log R\frac{\partial\theta}{\partial v}=0\,.$$

It is interesting to note that the determination of all surfaces belonging to an asymptotic complex is of the same nature as the problem of infinitesimal deformation of surfaces in ordinary space (see Darboux, "Théorie des Surfaces," Vol. IV, Ch. II).

We shall apply the above method to a few examples:

1°. Let the surface in M_5 be given in the form

$$x_1=\lambda_1(u)+\mu_1(v)\,,\quad x_2=\lambda_2(u)+\mu_2(v)=c\,(u-v)\,,$$
$$x_3=\lambda_3(u)+\mu_3(v)\,,\quad x_4=\lambda_4(u)+\mu_4(v)\,,\quad x_5=\lambda_5(u)+\mu_5(v)\,;$$

which is a two-dimensional translation surface.

The curves (u) and (v) are two sets of parallel curves. The condition (10) becomes, on substituting,

$$\mu_3'=\lambda_3'\,,$$

which can only be possible if $\mu_3=c_1v$ and $\lambda_3=c_1u$, so that $x_3=c_1(u+v)$. From 8 (b) we derive

$$\lambda_4'=-\frac{c}{c_1}\lambda_1',\quad \mu_4'=\frac{c}{c_1}\mu_1'\,,$$

hence, we must put $x_4 = -\frac{c}{c_1}(\lambda_1 - \mu_1)$. x_5 may now be calculated from 8 (a). We find

$$x_5 = 4c\int \lambda_1 du - 2cu\lambda_1 - 4c\int \mu_1 dv + 2cv\mu_1 .$$

We now put

$$4c\int \lambda_1 du = F, \quad -4c\int \mu_1 dv = F_1,$$

and also $c_1 = \frac{1}{4}$, $c = \frac{1}{2k}$ and our surface has now the form

$$\begin{aligned}
x_1 &= \frac{2}{k}(F' - F_1'), \\
x_2 &= \frac{u - v}{2k}, \\
x_3 &= \frac{u + v}{4}, \\
x_4 &= -F' - F_1', \\
x_5 &= F + F_1 - \frac{u}{2}F' - \frac{u}{2}F_1';
\end{aligned}$$

corresponding to this surface, we have in M_3

$$\begin{aligned}
X_1 &= \frac{u - v}{2k}, \\
X_2 &= -F' - F_1', \\
X_3 &= F + F_1 - \frac{u + v}{2}F' - \frac{u + v}{2}F_1',
\end{aligned}$$

a family of surfaces that has been obtained by Darboux (Leçons, Vol. I, p. 141) by an entirely different method; in fact, we only need to change the coordinate-system by means of the transformation

$$\frac{u - v}{2k} = \alpha, \quad \frac{u + v}{2} = \beta,$$

in order to get the identical form given by Darboux.

2°. Let $\phi_3 = uv$. We find from (10) $\phi_2 = \phi_2\left(\frac{u}{v}\right)$.

Substituting in (11) we get

$$\frac{\partial\phi_1}{\partial u} = \frac{-v^2\frac{\partial\phi_4}{\partial u}}{\phi_2'}, \quad \frac{\partial\phi_1}{\partial v} = \frac{v^2\frac{\partial\phi_4}{\partial v}}{\phi_2'}, \quad R = \frac{v^2}{\phi_2'}.$$

The differential equation (12) becomes

$$\frac{\partial^2\theta}{\partial u\,\partial v} + A\frac{\partial\theta}{\partial u} + B\frac{\partial\theta}{\partial v} = 0, \tag{13}$$

where
$$A = \frac{2v\phi_2' + u\phi_2''}{2v^2\phi_2'}, \quad B = \frac{-\phi_2''}{2v\phi_2'}.$$

We may now make the special hypothesis that the invariant

$$k = \frac{\partial B}{\partial v} + AB = 0,$$

from which we obtain on integrating

$$\phi_2 = c_1\frac{v}{u} + c_2\frac{u}{v} + c_3.$$

Substituting this value in A and B and integrating the differential equation (13) we get

$$\theta = \phi_4 = e^{-\int B du}\left(\sigma(v) + \int\rho(u)\, e^{\int B du - \int A dv}\, du\right).$$

ϕ_1 and ϕ_5 may now be calculated from the equations (11) and (8). That ϕ_2 also satisfies (13) may be verified by substituting in (13) for θ the function

$$c_1\frac{v}{u} + c_2\frac{u}{v} + c_3.$$

From equation (11) we have

$$\frac{\partial\phi_2}{\partial u} = \frac{1}{R}\cdot\frac{\partial\phi_3}{\partial u}, \quad \frac{\partial\phi_2}{\partial v} = -\frac{1}{R}\frac{\partial\phi_3}{\partial v},$$
$$\frac{\partial\phi_4}{\partial u} = -\frac{1}{R}\frac{\partial\phi_1}{\partial u}, \quad \frac{\partial\phi_4}{\partial v} = \frac{1}{R}\frac{\partial\phi_1}{\partial v},$$

from which we easily derive the following differential equation similar to the equation (12)

$$\frac{\partial^2\theta}{\partial u\,\partial v} - \tfrac{1}{2}\frac{\partial}{\partial v}\log R\,\frac{\partial\theta}{\partial u} - \tfrac{1}{2}\frac{\partial}{\partial u}\log R\,\frac{\partial\theta}{\partial v} = 0, \tag{14}$$

of which ϕ_1 and ϕ_3 are particular solutions; if two such solutions can be formed ϕ_2 and ϕ_4 may be obtained from equations (11) and ϕ_5 from (8). Since it is an equation with equal invariants it may be put in the form

$$\frac{\partial^2\bar\theta}{\partial u\,\partial v} = h_2\bar\theta, \tag{14'}$$

where

$$h_2 = -\tfrac{1}{2}\frac{\partial^2 \log R}{\partial u\,\partial v} + \tfrac{1}{4}\frac{\partial}{\partial u}\log R\,.\,\frac{\partial}{\partial u}\log R.$$

The particular solutions of (14′) are

$$\bar\theta_1 = \sqrt{R}, \quad \bar\theta_2 = \phi_1\sqrt{R}, \quad \bar\theta_3 = \phi_3\sqrt{R}.$$

If then three particular solutions of (14′) *can be found, we know how to obtain a surface belonging to the asymptotic complex.*

Example. Let $h_2 = 0$; we have then

$$\bar\theta = \rho(u) + \sigma(v);$$

we may therefore put

$$\bar\theta_1 = \rho(u) + \sigma(v), \quad \bar\theta_2 = \phi_1(\rho+\sigma) = (\rho_1+\sigma_1)(\rho+\sigma),$$
$$\bar\theta_3 = \phi_3(\rho+\sigma) = (\rho_3+\sigma_3)(\rho+\sigma),$$

so that we get

$$\phi_1 = \rho_1 + \sigma_1, \quad \phi_3 = \rho_3 + \sigma_3.$$

We also find

$$h_1 = \frac{\rho'\sigma'}{(\rho+\sigma)^2}.$$

The equation (12′) therefore becomes

$$\frac{\partial^2\bar\theta}{\partial u\,\partial v} = \frac{\rho'\sigma'}{(\rho+\sigma)^2}\bar\theta.$$

which may be integrated by Laplace's method. We find, taking two particular solutions,

$$\bar{\theta}_2 = -2\frac{\rho_2+\sigma_2}{\rho+\sigma} + \frac{\rho_2'}{\rho'} + \frac{\sigma_2'}{\sigma'},$$
$$\bar{\theta}_4 = -2\frac{\rho_4+\sigma_4}{\rho+\sigma} + \frac{\rho_4'}{\rho'} + \frac{\sigma_4'}{\sigma'}.$$

but $\bar{\theta}_2 = \frac{\phi_2}{\sqrt{R}}$ and $\bar{\theta}_4 = \frac{\phi_4}{\sqrt{R}}$, hence we get

$$\phi_2 = -2(\rho_2+\sigma_2) + \left(\frac{\rho_2'}{\rho'} + \frac{\sigma_2'}{\sigma'}\right)(\rho+\sigma),$$
$$\phi_4 = -2(\rho_4+\sigma_4) + \left(\frac{\rho_4'}{\rho'} + \frac{\sigma_4'}{\sigma'}\right)(\rho+\sigma).$$

ϕ_5 may now be obtained from (8) without difficulty, since we know that $d\phi_5$ is a perfect differential. We shall not reproduce the calculations here.

We may state the chief result of the preceding development thus:

There exists a one-to-one correspondence between all the surfaces of three-dimensional space and all the two-dimensional surfaces of five-dimensional space belonging to an asymptotic complex.

Given a surface in M_3 referred to its asymptotic curves, three particular solutions of (14′) are known and the corresponding surface in M_5 may be found. The geometrical meaning of R is obvious,*

$$R^2 = s^2 - rt,$$

so that if K denotes the total curvature of the given surface, we have

$$K = \frac{-R^2}{(1+P_1^2+P_2^2)^2}.$$

As an example, we may take a sphere $X_1^2+X_2^2+X_3^2=1$ referred to its rectilinear generators which we know are asymptotic lines. We have then

$$X_1 = \frac{1-\alpha\beta}{\alpha-\beta}, \quad X_2 = i\,\frac{1+\alpha\beta}{\alpha-\beta}, \quad X_3 = \frac{\alpha+\beta}{\alpha-\beta}.$$

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* See Darboux, Vol. IV, p. 21, where the quantity λ corresponds to the reciprocal of R.

Calculating P_1 and P_2, we find

$$P_1 = \frac{\alpha\beta - 1}{\alpha + \beta}, \quad P_2 = -i\frac{\alpha\beta + 1}{\alpha + \beta}.$$

The corresponding surface in M_5 is

$$x_1 = \tfrac{1}{2}\,\frac{\alpha\beta - 1}{\alpha + \beta}, \quad x_2 = \frac{1 - \alpha\beta}{\alpha - \beta}, \quad x_3 = -\frac{i}{2}\,\frac{1 + \alpha\beta}{\alpha + \beta},$$
$$x_4 = i\,\frac{1 + \alpha\beta}{\alpha - \beta}, \quad x_5 = \frac{\alpha^2 + \beta_2}{\alpha^2 - \beta^2}.$$

Eliminating α and β, we have the surface

$$\left.\begin{aligned} &(4x_1^2 + 4x_3^2 + 1)(1 - x_2^2 - x_4^2) = 1, \\ &x_2^2 + x_4^2 + (x_5 + x_1x_2 + x_3x_4)^2 = 1, \\ &2x_1x_5 + 2x_1^2x_2 + 2x_3^2x_2 + x_2 = 0, \end{aligned}\right\} \tag{15}$$

which satisfies the differential equations

$$\begin{aligned} &dx_5 + x_2dx_1 - x_1dx_2 + x_4dx_3 - x_3dx_4 = 0, \\ &dx_1dx_2 + dx_3dx_4 = 0. \end{aligned}$$

The lines (α) and (β) are curves corresponding to the rectilinear generators of the sphere; the surface may be considered as the envelope of ∞^2 straight lines belonging to the asymptotic complex.

Given in M_3 a ruled surface

$$\left.\begin{aligned} X_1 &= \phi_2 + \psi_2 v, \\ X_2 &= \phi_4 + \psi_4 v, \\ X_3 &= \phi + \xi v, \end{aligned}\right\} \tag{16}$$

to it there will correspond in M_5 a surface on which the lines (u) belong to an asymptotic complex while the lines (v) in general will be lines of the nullsystem. When will this surface be a ruled surface? Calculating P_1 and P_2 we find

$$P_1 = \frac{\phi'\psi_4 - \xi\phi_4' + (\psi_4\xi' - \psi_4'\xi)v}{\phi_2'\psi_4 - \phi_4'\psi_2 + (\psi_2'\psi_4 - \psi_4'\psi_2)v},$$
$$P_2 = \frac{-\phi'\psi_2 + \xi\phi_2' + (\psi_2'\xi - \psi_2\xi')v}{\phi_2'\psi_4 - \phi_4'\psi_2 + (\psi_2'\psi_4 - \psi_4'\psi_2)v},$$

and the surface in M_5 may be written

$$\left.\begin{aligned} &x_1 = \frac{P_1}{2}, \quad x_2 = \phi_2 + \psi_2 v, \quad x_3 = \frac{P_2}{2}, \quad x_4 = \phi_4 + \psi_4 v, \\ &x_5 = \phi = \xi v - \frac{P_1}{2}(\phi_2 + \psi_2 v) - \frac{P_2}{2}(\phi_4 + \psi_4 v), \end{aligned}\right\} \tag{16'}$$

which will be a ruled surface whenever the coefficient of v^2 on the right hand side of the last equation vanishes, that is, whenever $\psi_2'\psi_4 - \psi_4'\psi_2 = 0$, or $\psi_2 = c\psi_4$. Interpreted geometrically this means that the rectilinear generators are always parallel to a fixed plane which is perpendicular to the X_1X_2-plane; that is, we have all the ruled surfaces with a plane director.*

If we choose the functions ψ_4 and ξ in such a way that $c^2\psi_4^2 + \psi_4^2 + \xi^2 = 1$, v will denote the distance of any point on the rectilinear generator from the point where it intersects the fixed curve $v = 0$ and u will denote the angle which this line makes with its projection on the X_1X_2-plane. The surface then takes the form

$$\left.\begin{aligned} X_1 &= \phi_2 + \frac{c}{\sqrt{c^2+1}} \cos u \,.\, v, \\ X_2 &= \phi_4 + \frac{\cos u}{\sqrt{c^2+1}} \,.\, v, \\ X_3 &= \phi \;+ \sin u \,.\, v. \end{aligned}\right\} \tag{17}$$

If the director-plane is transformed into the X_2X_3-plane, c will be zero and the above equations become

$$X_1 = \phi_2, \quad X_2 = \phi_4 + \cos u \,.\, v, \quad X_3 = \phi + \sin u \,.\, v. \tag{17'}$$

As is well known, to this class of surfaces belong the screw-surfaces and conoids.† If the surface is a developable surface, it is cylindrical. If the rectilinear generators belong to the nullsystem

$$X_1 dX_2 - X_2 dX_1 + k dX_3 = 0,$$

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* These surfaces have been studied by Catalan, J. É.c. Polyt. 17 (1843), p. 121. See also Encyclopaedie der Mathematischen Wissenschaften, Band III$_3$, p. 271.

† Thus, if in (17′), $\phi_2 = au$, $\phi_4 = 0$, $\phi = 0$, we get the skew helicoid with plane director.

it is a developable surface and hence cylindrical. The proof of these statements we leave to the reader.

Conversely, *to every ruled surface in M_5 whose rectilinear generators (u) belong to an asymptotic complex and whose coordinate lines (v) belong to the nullsystem, there corresponds in M_3 a ruled surface with a plane-director perpendicular to the X_1X_2-plane.*

Let the surface be represented by the equations

$$x_i = \lambda_i(u) + \rho_i(u)\,v. \qquad i = 1,\ 2,\ 3,\ 4,\ 5 \tag{18}$$

The following conditions must be satisfied:

$$\left.\begin{aligned}
&\rho_5 = \rho_2\lambda_1 - \rho_1\lambda_2 + \rho_4\lambda_3 - \rho_3\lambda_4,\\
&\lambda_5' = \lambda_1\lambda_2' - \lambda_2\lambda_1' + \lambda_3\lambda_4' - \lambda_4\lambda_3',\\
&\rho_1\rho_2 + \rho_3\rho_4 = 0,\\
&\rho_2\rho_1' - \rho_1\rho_2' + \rho_3'\rho_4 - \rho_3\rho_4' = 0,\\
&\rho_5' + \rho_2\lambda_1' + \rho_1'\lambda_2 - \rho_1\lambda_2' - \lambda_1\rho_2' + \lambda_4\rho_3' + \lambda_3'\rho_4 - \lambda_4'\rho_3 - \lambda_3\rho_4' = 0.
\end{aligned}\right\} \tag{19}$$

These conditions may be further simplified. We differentiate the first and substitute the value of ρ_5' thus found in the last equation which reduces to

$$\lambda_2'\rho_1 - \lambda_1'\rho_2 + \lambda_4'\rho_3 - \lambda_3'\rho_4 = 0. \tag{20}$$

Differentiating the third and adding to the fourth, we obtain

$$\rho_2\rho_1' + \rho_4\rho_3' = 0. \tag{21}$$

From the third of equations (19) and from (21) we obtain

$$\frac{\rho_2}{\rho_4} = -\frac{\rho_3}{\rho_1} = -\frac{\rho_3'}{\rho_1'},$$

which is satisfied by putting

$$\rho_3 = -c\rho_1, \quad \rho_2 = c\rho_4.$$

Equation (20) now becomes

$$\lambda_2' - c\lambda_4' = \frac{\rho_4}{\rho_1}(c\lambda_1' + \lambda_3'),$$

which determines λ_3; ρ_5 and λ_5 may now be obtained from the first of equations (19). The surface in M_3 is

$$X_1 = \lambda_2 + c\rho_4 v\,,$$
$$X_2 = \lambda_4 + \rho_4 v\,,$$
$$X_3 = \lambda_5 + \lambda_1\lambda_2 + \lambda_3\lambda_4 + 2\rho_4\,(c\lambda_1 + \lambda_3)\,v\,,$$

which is a ruled surface whose plane director is perpendicular to the X_1X_2-plane.

We shall now prove the following

THEOREM.—*The second set of asymptotic lines of all ruled surfaces with a plane director may be obtained by quadratures.*

Choose for X_1X_2-plane any plane perpendicular to the plane director; the equations at the surface will be of the form

$$X_1 = \phi_2 + c\psi_4 v\,, \quad X_2 = \phi_4 + \psi_4 v\,, \quad X_3 = \phi + \xi v. \tag{22}$$

Calculating P_1 and P_2 we find,

$$\left.\begin{aligned} P_1 &= \frac{\phi'\psi_4 - \xi\phi_4' + \psi_4\,(\xi' - \xi)\,v}{\psi_4\,(\phi_2' - c\phi_4')} = \lambda_1 + \rho_1 v\,,\\ P_2 &= \frac{-c\phi'\psi_4 + \xi\phi_2' - c\psi_4\,(\xi' - \xi)\,v}{\psi_4\,(\phi_2' - c\phi_4')} = \lambda_3 - c\rho_1 v\,.\end{aligned}\right\} \tag{22'}$$

The differential equation of the asymptotic lines is

$$dX_1dP_1 + dX_2dP_2 = 0\,.$$

Substituting in this equation the values of dX_1, dX_2, dP_1, dP_2 obtained by differentiating equations (22) and (22′) we get an equation of the form

$$\frac{dv}{du} = A + Bv\,, \tag{23}$$

which is integrable by quadratures. Q. E. D.

From this theorem it follows that the second set of lines belonging to an asymptotic complex on the given ruled surface in M_5 of the form (13), the ρ's and λ's satisfying the conditions (19), can be obtained by quadratures. The differential equation which determines this system is

$$\frac{dv}{du} = \frac{\lambda_1'\lambda_2' + \lambda_3'\lambda_4' + \dfrac{\rho_1'\rho_4 + \rho_4'\rho_1}{\rho_1}\,(\lambda_3' + c\lambda_1')\,cv}{-\,2\rho_4\,(\lambda_3' + c\lambda_1')}\,;$$

which is of the form (23).

III.

A point-transformation in M_5 will as a rule transform the ∞^7 lines of the null-system into curves; if these curves are to be straight lines, the transformation must be projective; in fact, a line is defined by four linear spaces and a transformation that transforms a linear space into a linear space must necessarily be projective. There are in M_5 ∞^{35} such transformations defining the projective group of that space. Among all these transformations we shall consider those that transform any line of the nullsystem into some other line of the same system. These transformations define a group which we shall call with Lie *the projective group of the nullsystem.** This group is made up of the following 21 separate transformations:

$$\left.\begin{array}{l}
x_1\dfrac{\partial f}{\partial x_1}-x_2\dfrac{\partial f}{\partial x_2},\quad x_3\dfrac{\partial f}{\partial x_2}+x_1\dfrac{\partial f}{\partial x_4},\quad \dfrac{\partial f}{\partial x_5},\quad x_2\dfrac{\partial f}{\partial x_1},\quad x_1\dfrac{\partial f}{\partial x_2},\quad x_4\dfrac{\partial f}{\partial x_3},\quad x_3\dfrac{\partial f}{\partial x_4},\\
x_1\dfrac{\partial f}{\partial x_3}-x_4\dfrac{\partial f}{\partial x_4},\quad x_4\dfrac{\partial f}{\partial x_1}+x_2\dfrac{\partial f}{\partial x_3},\quad \dfrac{\partial f}{\partial x_2}-x_1\dfrac{\partial f}{\partial x_5},\quad \dfrac{\partial f}{\partial x_4}-x_3\dfrac{\partial f}{\partial x_5},\\
x_4\dfrac{\partial f}{\partial x_3}-x_3\dfrac{\partial f}{\partial x_4},\quad \dfrac{\partial f}{\partial x_1}+x_2\dfrac{\partial f}{\partial x_5},\quad \dfrac{\partial f}{\partial x_3}+x_4\dfrac{\partial f}{\partial x_5},\quad x_2\dfrac{\partial f}{\partial x_4}-x_3\dfrac{\partial f}{\partial x_1},\\
x_5\dfrac{\partial f}{\partial x_2}-x_2\sum_1^5 x_i\dfrac{\partial f}{\partial x_i},\quad x_5\dfrac{\partial f}{\partial x_1}+x_1\sum_1^5 x_i\dfrac{\partial f}{\partial x_i},\quad x_5\dfrac{\partial f}{\partial x_4}-x_4\sum_1^5 x_i\dfrac{\partial f}{\partial x_i},\\
x_5\dfrac{\partial f}{\partial x_3}+x_3\sum_1^5 x_i\dfrac{\partial f}{\partial x_i},\quad x_5\dfrac{\partial f}{\partial x_5}+\sum_1^5 x_i\dfrac{\partial f}{\partial x_i},\quad x_5\sum x_i\dfrac{\partial f}{\partial x_i}.
\end{array}\right\}\quad(1)$$

But to any point-transformation in M_5 leaving the nullsystem invariant there corresponds in M_3 a contact-transformation. In fact, from the invariance of the Pfaffian equation

$$dx_5+x_2dx_1-x_1dx_2+x_4dx_3-x_3dx_4=0,$$

there follows by virtue of the transformation

$$x_1=\frac{P_1}{2},\quad x_2=X_1,\quad x_3=\frac{P_2}{2},\quad x_4=X_2,\quad x_5+x_1x_2+x_3x_4=X_3 \qquad (2)$$

also the invariance of

$$dX_3-P_1dX_1-P_2dX_2=0,$$

which means that the transformation in M_3 is a contact-transformation. The group of contact-transformations may be found by substituting in (1) the new variable from (2). We find then the group

* See Lie, Theorie der Transformationsgruppen, Ab. II, p. 521.

$$\left.\begin{aligned}
&P_1\frac{\partial f}{\partial P_1}-X_1\frac{\partial f}{\partial X_1},\quad X_2\frac{\partial f}{\partial X_2}-P_2\frac{\partial f}{\partial P_2},\quad X_1\frac{\partial f}{\partial X_3}+\frac{\partial f}{\partial P_1},\quad X_2\frac{\partial f}{\partial X_3}+\frac{\partial f}{\partial P_2},\\
&P_1\frac{\partial f}{\partial P_2}-X_2\frac{\partial f}{\partial X_1},\quad X_1\frac{\partial f}{\partial X_2}-P_2\frac{\partial f}{\partial P_1},\quad \frac{\partial f}{\partial X_1},\\
&\qquad\qquad\qquad\qquad\qquad\frac{\partial f}{\partial X_2},\quad \frac{\partial f}{\partial X_3},\quad P_2^2\frac{\partial f}{\partial X_3}+2P_2\frac{\partial f}{\partial X_2}\\
&2X_3\frac{\partial f}{\partial X_3}+P_1\frac{\partial f}{\partial P_1}+P_2\frac{\partial f}{\partial P_2}+X_1\frac{\partial f}{\partial X_1}+X_2\frac{\partial f}{\partial X_2},\\
&\qquad P_1^2\frac{\partial f}{\partial X_3}+2P_1\frac{\partial f}{\partial X_1},\quad X_2\frac{\partial f}{\partial P_1}+X_1\frac{\partial f}{\partial P_2}+X_1X_2\frac{\partial f}{\partial X_3},\\
&X_2^2\frac{\partial f}{\partial X_3}+2X_2\frac{\partial f}{\partial P_2},\quad P_2\frac{\partial f}{\partial X_1}+P_1\frac{\partial f}{\partial X_2}+P_1P_2\frac{\partial f}{\partial X_3},\quad X_1^2\frac{\partial f}{\partial X_3}+2X_1\frac{\partial f}{\partial P_1},\\
&[X_3-\tfrac{1}{2}(P_1X_1+P_2X_2)]\frac{\partial f}{\partial P_1}+\tfrac{1}{2}X_1\Big[P_1\frac{\partial f}{\partial P_1}\\
&\qquad\qquad+P_2\frac{\partial f}{\partial P_2}+\sum_1^3 X_i\frac{\partial f}{\partial X_i}+X_3\frac{\partial f}{\partial X_3}\Big],\\
&[X_3-\tfrac{1}{2}(P_1X_1+P_2X_2)]\frac{\partial f}{\partial P_2}+\tfrac{1}{2}X_2\Big[P_1\frac{\partial f}{\partial P_1}\\
&\qquad\qquad+P_2\frac{\partial f}{\partial P_2}+\sum_1^3 X_i\frac{\partial f}{\partial X_i}+X_3\frac{\partial f}{\partial X_3}\Big],\\
&[X_3-\tfrac{1}{2}(P_1X_1+P_2X_2)]\Big(\frac{\partial f}{\partial X_1}+P_1\frac{\partial f}{\partial X_3}\Big)-\frac{P_1}{2}\Big[P_1\frac{\partial f}{\partial P_1}\\
&\qquad\qquad+P_2\frac{\partial f}{\partial P_2}+\sum X_i\frac{\partial f}{\partial X_i}+X_3\frac{\partial f}{\partial X_3}\Big],\\
&[X_3-\tfrac{1}{2}(P_1X_1+P_2X_2)]\Big(\frac{\partial f}{\partial X_1}+P_2\frac{\partial f}{\partial X_3}\Big)-\frac{P_2}{2}\Big[P_1\frac{\partial f}{\partial P_1}\\
&\qquad\qquad+P_2\frac{\partial f}{\partial P_2}+\sum X_i\frac{\partial f}{\partial X_i}+X_3\frac{\partial f}{\partial X_3}\Big],\\
&[X_3-\tfrac{1}{2}(P_1X_1+P_2X_2)]^2\frac{\partial f}{\partial X_3}-[X_3-\tfrac{1}{2}(P_1X_1+P_2X_2)]\Big[P_1\frac{\partial f}{\partial P_1}\\
&\qquad\qquad+P_2\frac{\partial f}{\partial P_2}+\sum X_i\frac{\partial f}{\partial X_i}+X_3\frac{\partial f}{\partial X_3}\Big].
\end{aligned}\right\}\quad(3)$$

This group transforms the ∞^5 parabolae and their surface-elements into themselves. But the parabolae are the integral curves of the differential equations

$$\frac{d^3X_3}{dX_1^3}=0,\quad \frac{d^2X_2}{dX_1^2}=0.$$

Hence we have the

THEOREM.—*There exists in ordinary space* ∞^{21} *contact-transformations which leave the differential equations*

$$\frac{d^3X_3}{dX_1^3} = 0, \quad \frac{d^2X_2}{dX_1^2} = 0 \tag{4}$$

invariant. Moreover, there are 21 *infinitesimal contact-transformations leaving these equations invariant.*

Of all the projective transformations that leave the nullsystem invariant, the Euclidian motion presents the greatest interest. That such a motion exists is already evident from the definition of a nullsystem according to which the line-elements of the system move perpendicular to the direction of the n-dimensional screw-motion defined by the equations

$$\frac{\delta x_1}{\delta t} = x_2, \quad \frac{\delta x_2}{\delta t} = -x_1, \quad \ldots\ldots, \quad \frac{\delta x_{u-4}}{\delta t} = x_{u-3}, \quad \frac{\delta x_{u-3}}{\delta t} = -x_{u-4},$$

$$\frac{\delta x_3}{\delta t} = x_4, \quad \frac{\delta x_4}{\delta t} = -x_3, \quad \ldots\ldots, \quad \frac{\delta x_{u-2}}{\delta t} = x_{u-1}, \quad \frac{\delta x_{u-1}}{\delta t} = -x_{u-2},$$

$$\frac{\delta x_u}{\delta t} = c_u.$$

The system must therefore remain invariant during this motion.

Are there other Euclidian motions of the same nature? To answer this question we shall first consider the case $n = 5$ and construct the group of Euclidian motions, leaving the nullsystem of M_5 invariant. A Euclidian transformation group is defined by the system of equations

$$\left.\begin{aligned}
\frac{\delta x_1}{\delta t} &= \alpha_{12}x_2 + \alpha_{13}x_3 + \alpha_{14}x_4 + \alpha_{15}x_5 + c_1,\\
\frac{\delta x_2}{\delta t} &= -\alpha_{12}x_1 + \alpha_{23}x_3 + \alpha_{24}x_4 + \alpha_{25}x_5 + c_2,\\
\frac{\delta x_3}{\delta t} &= -\alpha_{13}x_1 - \alpha_{23}x_2 + \alpha_{34}x_4 + \alpha_{35}x_5 + c_3,\\
\frac{\delta x_4}{\delta t} &= -\alpha_{14}x_1 - \alpha_{24}x_2 - \alpha_{34}x_3 + \alpha_{45}x_5 + c_4,\\
\frac{\delta x_5}{\delta t} &= -\alpha_{15}x_1 - \alpha_{25}x_2 - \alpha_{35}x_3 - \alpha_{45}x_4 + c_5.
\end{aligned}\right\} \tag{5}$$

Since this transformation is to leave the nullsystem invariant, we must have

$$\delta\left(dx_5 + x_2 dx_1 - x_1 dx_2 + x_4 dx_3 - x_3 dx_4\right) \equiv 0,$$

or, what is the same thing,

$$d\delta x_5 + \delta x_2 dx_1 + x_2 d\delta x_1 \ldots \equiv 0.$$

Substituting in this equation the values of δx_i taken from (5) and equating to zero the coefficients of $x_i dx_k$ and dx_i, $(k, i = 1, 2, 3, 4, 5)$, we obtain

$$\begin{aligned} &\alpha_{15} = \alpha_{25} = \alpha_{35} = \alpha_{45} = c_1 = c_2 = c_3 = c_4 = 0, \\ &\alpha_{13} - \alpha_{24} = 0, \quad \alpha_{14} + \alpha_{23} = 0, \end{aligned}$$

so that the system (5) may now be written

$$\left.\begin{aligned} \frac{\delta x_1}{\delta t} &= \alpha_{12}x_2 + \alpha_{13}x_3 + \alpha_{14}x_4, \\ \frac{\delta x_2}{\delta t} &= -\alpha_{12}x_1 - \alpha_{14}x_3 + \alpha_{13}x_4, \\ \frac{\delta x_3}{\delta t} &= -\alpha_{13}x_1 + \alpha_{14}x_2 + \alpha_{34}x_4, \\ \frac{\delta x_4}{\delta t} &= -\alpha_{14}x_1 - \alpha_{14}x_2 + \alpha_{34}x_3, \\ \frac{\delta x_5}{\delta t} &= c_5, \end{aligned}\right\} \tag{5'}$$

which shows that the group is composed of the following independent infinitesimal transformations

$$\begin{gathered} x_2\frac{\partial f}{\partial x_1} - x_1\frac{\partial f}{\partial x_2}, \quad x_4\frac{\partial f}{\partial x_3} - x_3\frac{\partial f}{\partial x_4}, \quad \frac{\partial f}{\partial x_5}, \\ x_3\frac{\partial f}{\partial x_1} + x_4\frac{\partial f}{\partial x_2} - x_1\frac{\partial f}{\partial x_3} - x_2\frac{\partial f}{\partial x_4}, \quad x_4\frac{\partial f}{\partial x_1} - x_3\frac{\partial f}{\partial x_2} + x_2\frac{\partial f}{\partial x_3} - x_1\frac{\partial f}{\partial x_4}, \end{gathered} \tag{6}$$

that is to say *four rotations* and a translation along the x_5 — axis. It is not at all difficult to extend this method to any odd number of dimensions. We obtain

after some lengthy calculations the following system of equations:

$$\frac{\delta x_1}{\delta t} = \alpha_{12}x_2 + \alpha_{13}x_3 + \alpha_{14}x_4 + \alpha_{15}x_5 + \alpha_{16}x_6 + \dots + \alpha_{1n-1}x_{n-1},$$

$$\frac{\delta x_2}{\delta t} = -\alpha_{12}x_1 - \alpha_{14}x_3 + \alpha_{13}x_4 + \alpha_{16}x_5 + \alpha_{15}x_6 + \dots + \alpha_{1n-2}x_{n-1},$$

$$\frac{\delta x_3}{\delta t} = -\alpha_{12}x_1 + \alpha_{14}x_2 + \alpha_{34}x_4 + \alpha_{35}x_5 + \alpha_{36}x_6 + \dots + \alpha_{3n-1}x_{n-1},$$

$$\frac{\delta x_4}{\delta t} = -\alpha_{14}x_1 - \alpha_{13}x_2 - \alpha_{34}x_3 - \alpha_{36}x_5 + \alpha_{35}x_6 - \dots + \alpha_{3n-2}x_{n-1},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\frac{\delta x_{n-1}}{\delta t} = -\alpha_{1n-1}x_1 - \alpha_{1n-2}x_2 - \alpha_{3n-1}x_3 - \dots - \alpha_{n-2,n-2}x_{n-1},$$

$$\frac{\delta x_n}{\delta t} = c_n,$$

from which we obtain a group containing $\left(\frac{n-1}{2}\right)^2$ rotations and a translation along the x_n-axis. Hence the

THEOREM.—*The most general Euclidian motion in M_5, leaving the nullsystem*

$$dx_n + \sum_{i=1}^{i=\frac{n-1}{2}} (x_{2i}dx_{2i-1} - x_{2i-1}dx_{2i}) = 0$$

invariant, is made up of $\left(\frac{n-1}{2}\right)^2$ rotations around the origin and a translation along the x_n-axis. Among these $\left(\frac{n-1}{2}\right)^2$ rotations are also included the $\frac{n-1}{2}$ rotations of the n-dimensional screw-motion.

For $n = 5$ we can easily deduce the finite equations of the transformation group (6). If we extend the method described in Lie-Scheffer's "Vorlesungen über Continuierliche Gruppen," p. 196, to the case of 5 variables, we obtain the equations

$$\begin{aligned}
\bar{x}_1 &= x_1 (\cos t_1 \cos t_2 \cos t_3 + \sin t_1 \sin t_2 \sin t_3) \\
&\quad + x_2 (\sin t_1 \cos t_2 \cos t_3 - \sin t_1 \sin t_2 \sin t_3) + x_3 \sin t_2 \cos t_3 + x_4 \cos t_2 \sin t_3 , \\
\bar{x}_2 &= x_1 (\cos t_1 \sin t_2 \sin t_3 - \sin t_1 \cos t_2 \cos t_3) \\
&\quad + x_2 (\cos t_1 \cos t_2 \cos t_3 + \sin t_1 \sin t_2 \sin t_3) - x_3 \cos t_2 \sin t_3 + x_4 \sin t_2 \cos t_3, \\
\bar{x}_3 &= - x_1 [\sin t_3 \cos t_2 \sin (t_1 + t_4) + \sin t_2 \cos t_3 \cos (t_1 + t_4)] \\
&\qquad + x_2 [\cos t_2 \sin t_3 \cos (t_1 + t_4) - \sin t_2 \cos t_3 \sin (t_1 + t_4)] \\
&\qquad + x_3 (\cos t_2 \cos t_3 \cos t_4 - \sin t_2 \sin t_3 \sin t_4) \\
&\qquad + x_4 (\cos t_2 \cos t_3 \sin t_4 + \sin t_2 \sin t_3 \cos t_4), \\
\bar{x}_4 &= - x_1 [\cos t_2 \sin t_3 \cos (t_1 + t_4) - \sin t_2 \cos t_3 \sin (t_1 + t_4)] \\
&\qquad - x_2 [\cos t_3 \sin t_2 \cos (t_1 + t_4) + \cos t_2 \sin t_3 \sin (t_1 + t_4)] \\
&\qquad - x_3 (\cos t_2 \cos t_3 \sin t_4 + \sin t_2 \sin t_3 \cos t_4) \\
&\qquad + x_4 (\cos t_2 \cos t_3 \cos t_4 - \sin t_3 \sin t_4 \sin t_2), \\
\bar{x}_5 &= x_5 + t_5 .
\end{aligned}$$

The separate transformations of the group may now be obtained by putting

$$t_2 = t_3 = t_4 = t_5 = 0 ; \quad t_1 = t_2 = t_3 = t_5 = 0 ; \quad t_1 = t_3 = t_4 = t_5 = 0 ;$$
$$t_1 = t_2 = t_4 = t_5 = 0 ; \quad t_1 = t_2 = t_3 = t_4 = 0$$

in succession; we get

$$\text{(a)} \begin{cases} \bar{x}_1 = x_1 \cos t_1 + x_2 \sin t_1, \\ \bar{x}_2 = - x_1 \sin t_1 + x_2 \cos t_1, \\ \bar{x}_3 = x_3, \\ \bar{x}_4 = x_4, \\ \bar{x}_5 = x_5, \end{cases} \qquad \text{(b)} \begin{cases} \bar{x}_1 = x_1, \\ \bar{x}_2 = x_2, \\ \bar{x}_3 = x_3 \cos t_4 + x_4 \sin t_4, \\ \bar{x}_4 = - x_3 \sin t_4 + x_4 \cos t_4, \\ \bar{x}_5 = x_5, \end{cases}$$

$$\text{(c)} \begin{cases} \bar{x}_1 = x_1 \cos t_2 + x_3 \sin t_2, \\ \bar{x}_2 = x_2 \cos t_2 + x_4 \sin t_2, \\ \bar{x}_3 = - x_1 \sin t_2 + x_3 \cos t_2, \\ \bar{x}_4 = - x_2 \sin t_2 + x_4 \cos t_2, \\ \bar{x}_5 = x_5, \end{cases} \quad \text{(d)} \begin{cases} \bar{x}_1 = x_1 \cos t_3 + x_4 \sin t_3, \\ \bar{x}_2 = x_2 \cos t_3 - x_3 \sin t_3, \\ \bar{x}_3 = x_2 \sin t_3 + x_3 \cos t_3, \\ \bar{x}_4 = - x_1 \sin t_3 + x_4 \cos t_3, \\ \bar{x}_5 = x_5, \end{cases} \quad \text{(e)} \begin{cases} \bar{x}_1 = x_1, \\ \bar{x}_2 = x_2, \\ \bar{x}_3 = x_3, \\ \bar{x}_4 = x_4, \\ \bar{x}_5 = x_5 + t_5 . \end{cases}$$

Of the four rotations (a), (b), (c) and (d) the rotation (c) occupies a peculiar position in as much as *it is the only one that will make the nullsystem invariant in case it degenerates into an asymptotic complex,* that is to say, it is the only one that will render invariant the two differential equations

$$dx_5 + x_2 dx_1 - x_1 dx_2 + x_4 dx_3 - x_3 dx_4 = 0 ,$$
$$dx_1 dx_2 + dx_3 dx_4 = 0 .$$

The group represented by

$$Uf = x_3 \frac{\partial f}{\partial x_1} + x_4 \frac{\partial f}{\partial x_2} - x_1 \frac{\partial f}{\partial x_3} - x_2 \frac{\partial f}{\partial x_3}$$

is therefore characteristic of an asymptotic complex and expresses the fact that *such a complex has only one degree of mobility*, when we except the translation along the x_5-axis which is common to all the complexes that can be formed from a given non-special nullsystem in M_5.

The following question now suggests itself: What projective transformation will transform an asymptotic complex into itself? To attack this problem directly involves rather extensive formula work; but if we remember that the lines of the complex are transformed into all the lines of M_3 and their surface elements, the question resolves itself into finding all the contact-transformations which will transform these lines into themselves. Such a transformation must evidently transform, 1°, either plane into plane and point into point, or 2°, plane into point and point into plane; that is to say, it must be either a projective transformation or a dualistic transformation. We shall consider the former first. What must be the nature of this projective transformation? According to the method by which we obtained the ∞^4 lines of M_3 (see p. 123), any straight line may be considered as a degenerate parabola lying in a plane parallel to the X_3-axis. But each parabola is the intersection of this plane with a parabolic cylinder parallel to the X_2-axis; this cylinder is tangent to the plane at infinity. The required transformation must, therefore, transform parabolic cylinders into parabolic cylinders and consequently the plane at infinity into itself. It follows, then, that the transformation must be linear; furthermore, it must transform planes parallel to the X_3-axis into planes parallel to the same axis, so that finally we arrive at the following transformation:

$$\left.\begin{aligned} \overline{X}_1 &= a_1X_1 + b_1X_2 + d_1, \\ \overline{X}_2 &= a_2X_1 + b_2X_2 + d_2, \\ \overline{X}_3 &= a_3X_1 + b_3X_2 + c_3X_3 + d_3, \end{aligned}\right\} \tag{7}$$

$$\overline{P}_1 = \frac{b_2c_3}{a_1b_2 - a_2b_1} P_1 - \frac{a_2c_3}{a_1b_2 - a_2b_1} P_2 + \frac{a_3b_2 - a_2b_3}{a_1b_2 - a_2b_1},$$

$$\overline{P}_2 = \frac{-b_1c_3}{a_1b_2 - a_2b_1} P_1 + \frac{a_1c_3}{a_1b_2 - a_2b_1} P_2 + \frac{a_1b_3 - a_3b_1}{a_1b_2 - a_2b_1}.$$

The last two equations are obtained from the first three by extension* (ERWEITERUNG).

The corresponding group of infinitesimal contact-transformations contains 10 independent transformations

$$\frac{\partial f}{\partial X_1},\quad \frac{\partial f}{\partial X_2},\quad \frac{\partial f}{\partial X_3},\quad X_1\frac{\partial f}{\partial X_1}-P_1\frac{\partial f}{\partial P_1},\quad X_1\frac{\partial f}{\partial X_2}-P_2\frac{\partial f}{\partial P_1},\quad X_2\frac{\partial f}{\partial X_1}-P_1\frac{\partial f}{\partial X_2},$$

$$X_2\frac{\partial f}{\partial X_2}-P_2\frac{\partial f}{\partial P_2},\quad X_1\frac{\partial f}{\partial X_3}+\frac{\partial f}{\partial P_1},\quad X_2\frac{\partial f}{\partial X_3}+\frac{\partial f}{\partial P_2},\quad \sum_1^2\left(X_i\frac{\partial f}{\partial X_i}+P_i\frac{\partial f}{\partial P_i}\right)+2X_3\frac{\partial f}{\partial X_3}.$$

In the space M_5 we obtain a linear projective transformation

$$\left.\begin{aligned}
2\bar{x}_1 &= \frac{2b_2c_3}{a_1b_2-a_2b_1}x_1-\frac{2a_2c_3}{a_1b_2-b_1a_2}x_3+\frac{a_3b_2-a_2b_3}{a_1b_2-b_1a_2},\\
\bar{x}_2 &= a_1x_2+b_2x_4+d,\\
2\bar{x}_3 &= -\frac{2b_1c_3}{a_1b_2-a_2b_1}x_1+\frac{2a_1c_3}{a_1b_2-a_2b_1}x_3+\frac{a_1b_3-a_3b_1}{a_1b_2-b_1a_2},\\
\bar{x}_4 &= a_2x_2+b_2x_4+d_2,\\
\bar{x}_5 &= \frac{a_3}{2}x_2+\frac{b_3}{2}x_4+c_3x_5+\tfrac{1}{2}\left[\frac{d_1(a_3b_2-a_2b_3)+d_2(a_1b_3-a_3b_1)}{a_1b_2-b_1a_2}\right]+d_3,
\end{aligned}\right\}\quad (8)$$

which will give rise to an infinitesimal group of 10 independent transformations

$$x_1\frac{\partial f}{\partial x_1}-x_2\frac{\partial f}{\partial x_2},\quad x_2\frac{\partial f}{\partial x_4}-x_3\frac{\partial f}{\partial x_1}.\quad x_4\frac{\partial f}{\partial x_2}-x_1\frac{\partial f}{\partial x_4},\quad x_4\frac{\partial f}{\partial x_1}-x_3\frac{\partial f}{\partial x_3},$$

$$x_4\frac{\partial f}{\partial x_5}+\frac{\partial f}{\partial x_3},\quad x_2\frac{\partial f}{\partial x_5}+\frac{\partial f}{\partial x_1},\quad 2x_5\frac{\partial f}{\partial x_5}+\sum_1^4 x_i\frac{\partial f}{\partial x_i},$$

$$\frac{\partial f}{\partial x_2},\quad \frac{\partial f}{\partial x_4},\quad \frac{\partial f}{\partial x_5}.$$

We have thus found the following

THEOREM.—*There exist in the space M_5 ∞^{10} projective transformations for which an asymptotic complex remains invariant.*

All the transformations that transform the lines of an asymptotic complex into themselves are included in these ∞^{10} projective transformations and a

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* Lie, Theorie d. Transformationsgr., Ab. II, p. 46.

dualistic transformation. This latter transformation remains to be investigated. Since it must leave invariant the equation

$$dX_3 - P_1 dX_1 - P_2 dX_2 = 0$$

and also

$$dX_1 dP_1 + dX_2 dP_2 = 0\,,$$

it can be no other than the classic one of Euler's, viz.:

$$\left.\begin{array}{lll} \bar{X}_1 = P_2, & \bar{X}_2 = -P_1, & \bar{X}_3 = X_3 - X_1 P_1 - X_2 P_2, \\ \bar{P}_1 = -X_2, & & \bar{P}_2 = +X_1, \end{array}\right\} \tag{9}$$

corresponding to which we have in M_5 the transformation

$$\bar{x} = 2x_3, \quad \bar{x}_4 = -2x_1, \quad 2\bar{x}_1 = -x_4, \quad 2\bar{x}_3 = +x_2, \quad \bar{x}_5 = x_5, \tag{9'}$$

which leaves invariant the asymptotic complex. All transformations leaving the asymptotic complex invariant are thus made up of a combination of this transformation and any one of the group of ∞^{10} projective transformations (8).

The relation of Euler's transformation (9) to asymptotic complexes and its effect on asymptotic curves I shall discuss in another paper.

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